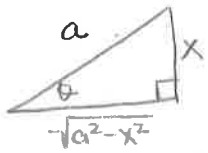
Trig Substitutions WS 3-19 odd

③ $\int_0^a \sqrt{a^2 - x^2}\, dx$

$x^2 = a^2 \sin^2\theta$

$x = a\sin\theta$

$dx = a\cos\theta\, d\theta$



$\sin\theta = \frac{x}{a}$ $\quad \theta = \sin^{-1}\frac{x}{a}$

$a^2 = x^2 + b^2$

$\sqrt{a^2 - x^2} = b$

$\int_0^a a\cos\theta \cdot a\cos\theta\, d\theta$

$a^2 \int_0^a \cos^2\theta\, d\theta$

$\frac{a^2}{2} \int_0^a 1 + \cos 2\theta\, d\theta$

$\frac{a^2}{2}\left(\theta + \frac{1}{2}\sin 2\theta\right)_0^a$

$\frac{a^2}{2}\left(\sin^{-1}\frac{x}{a} + \frac{1}{2}\cdot 2\sin\theta\cdot\cos\theta\right)_0^a$

$\frac{a^2}{2}\left(\sin^{-1}\frac{x}{a} + \frac{x}{a}\cdot\frac{\sqrt{a^2-x^2}}{a}\right)_0^a$

$\frac{a^2}{2}\left(\sin^{-1}\frac{x}{a} + \frac{x}{a^2}\sqrt{a^2-x^2}\right)_0^a$

Replace

$\sqrt{a^2 - a^2\sin^2\theta}$

$\sqrt{a^2(1-\sin^2\theta)}$

$\sqrt{a^2\cos^2\theta}$

$a\cos\theta$

$\rightarrow \frac{a^2}{2}\left(\left(\sin^{-1}(1) + \frac{a}{a^2}\cdot 0\right) - \left(\sin^{-1}0 + 0\right)\right)$

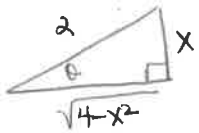
$\frac{a^2}{2}\left(\pi/2 - 0\right) = \frac{a^2\pi}{2\cdot 2}$ $\quad \frac{\pi a^2}{2^2} = \boxed{\frac{a^2\pi}{4}}$

⑦ $\int_0^1 \frac{dx}{\sqrt{4-x^2}}$

$x^2 = 4\sin^2\theta$

$x = 2\sin\theta$

$dx = 2\cos\theta\, d\theta$



$\sin\theta = \frac{x}{2}$ $\quad \theta = \sin^{-1}\frac{x}{2}$

$4 = x^2 + b^2$

$\sqrt{4-x^2} = b$

$\int_0^1 \frac{2\cos\theta\, d\theta}{2\cos\theta}$

Replace

$\sqrt{4-4\sin^2\theta}$

$\sqrt{4(1-\sin^2\theta)}$

$\sqrt{4\cos^2\theta}$

$2\cos\theta$

$\int_0^1 1\, d\theta$

$\theta\,|_0^1$

$\sin^{-1}\frac{x}{2}\,|_0^1$

$\sin^{-1}\frac{1}{2} - \sin^{-1}0$

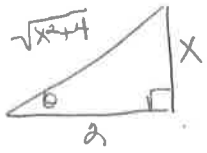
$\frac{\pi}{6} - 0 = \boxed{\frac{\pi}{6}}$

$\int_0^2 \frac{x\,dx}{4+x^2}$

$x^2 = 4\tan^2\theta$

$x = 2\tan\theta$

$dx = 2\sec^2\theta\,d\theta$



$\tan\theta = \frac{x}{2}$

$\sqrt{x^2+4} = r$

$\int_0^2 \frac{2\tan\theta \cdot 2\sec^2\theta\,d\theta}{4\sec^2\theta}$

Replace

$4 + 4\tan^2\theta$

$4(1+\tan^2\theta)$

$4\sec^2\theta$

$\int_0^2 \tan\theta\,d\theta$

$\ln|\sec\theta|$

$\ln\left|\frac{\sqrt{x^2+4}}{2}\right|_0^2$

$\ln\left|\frac{\sqrt{4+4}}{2}\right| - \ln\left|\frac{\sqrt{0+4}}{2}\right|$

$\ln\left|\frac{\sqrt{8}}{2}\right| - \ln\left|\frac{2}{2}\right|$

$\ln\left|\frac{\sqrt{8}}{2}\right| - 0 \quad (\ln 1)$

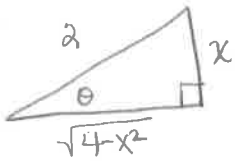
$= \ln\left|\frac{2\sqrt{2}}{2}\right| = \ln|\sqrt{2}| = \boxed{\frac{1}{2}\ln 2}$

$\int_0^1 \frac{dx}{4-x^2}$

$x^2 = 4\sin^2\theta$

$x = 2\sin\theta$

$dx = 2\cos\theta\,d\theta$



$\sin\theta = \frac{x}{2}$

$4 = x^2 + b^2$

$\sqrt{4-x^2} = b$

$\int_0^1 \frac{2\cos\theta\,d\theta}{4\cos^2\theta}$

Replace

$4 - 4\sin^2\theta$

$4(1-\sin^2\theta)$

$4\cos^2\theta$

$\frac{1}{2}\int_0^1 \frac{1}{\cos\theta}\,d\theta$

$\frac{1}{2}\int_0^1 \sec\theta\,d\theta$

$\frac{1}{2}\left[\ln|\sec\theta+\tan\theta|\right]_0^1$

$\frac{1}{2}\left[\ln\left|\frac{2}{\sqrt{4-x^2}} + \frac{x}{\sqrt{4-x^2}}\right|\right]_0^1$

$\frac{1}{2}\left[\ln\left|\frac{2}{\sqrt{3}}+\frac{1}{\sqrt{3}}\right| - \ln\left[\frac{2}{2}+\frac{0}{2}\right]\right]$

$\frac{1}{2}\left[\ln\left|\frac{3}{\sqrt{3}}\right| - \ln 1\right]$

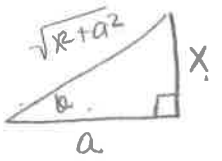
$\frac{1}{2}\ln|\sqrt{3}| = \boxed{\frac{1}{4}\ln 3}$

13) $\int \frac{dx}{x\sqrt{a^2+x^2}}$

$x^2 = a^2\tan^2\theta$

$x = a\tan\theta$

$dx = a\sec^2\theta\, d\theta$



$\tan\theta = \frac{x}{a}$

$r^2 = x^2 + a^2$

$\int \frac{a\sec^2\theta\, d\theta}{a\tan\theta \cdot a\sec\theta}$

Replace

$\sqrt{a^2+a^2\tan^2\theta}$

$\sqrt{a^2(1+\tan^2\theta)}$

$\sqrt{a^2\sec^2\theta}$

$a\sec\theta$

$\frac{1}{a}\int \frac{\sec\theta}{\tan\theta}\, d\theta$

$\frac{1}{a}\int \frac{\frac{1}{\cos\theta}}{\frac{\sin\theta}{\cos\theta}}\, d\theta$

$\frac{1}{a}\int \frac{1}{\sin\theta}\, d\theta$

$\frac{1}{a}\int \csc\theta\, d\theta$

$-\frac{1}{a}\ln|\csc\theta+\cot\theta|$

$-\frac{1}{a}\ln\left|\frac{\sqrt{x^2+a^2}}{x} - \frac{a}{x}\right|^{-1} = \frac{1}{a}\ln\left|\frac{\sqrt{x^2+a^2}+a}{x}\right| + C \rightarrow \boxed{\frac{1}{a}\ln\left|\frac{x}{a+\sqrt{x^2+a^2}}\right| + C}$

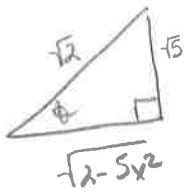
15) $\int \frac{dx}{\sqrt{2-5x^2}}$

$5x^2 = 2\sin^2\theta$

$\sqrt{5}\,x = \sqrt{2}\sin\theta$

$x = \sqrt{2/5}\sin\theta\, d\theta$

$dx = \sqrt{2/5}\cos\theta\, d\theta$



$\sin\theta = \frac{\sqrt{5}x}{\sqrt{2}}$

$\theta = \sin^{-1} x\sqrt{5/2}$

Replace

$\sqrt{2-2\sin^2\theta}$

$\sqrt{2(1-\sin^2\theta)}$

$\sqrt{2\cos^2\theta}$

$\sqrt{2}\cos\theta$

$\int \frac{\sqrt{\frac{2}{5}}\cos\theta\, d\theta}{\sqrt{2}\cos\theta}$

$\frac{\sqrt{\frac{2}{5}}}{\sqrt{2}}\int d\theta$

$\sqrt{\frac{2}{5}\cdot\frac{1}{2}} = \sqrt{\frac{1}{5}}\int d\theta$

$\frac{1}{\sqrt{5}}\theta + C$

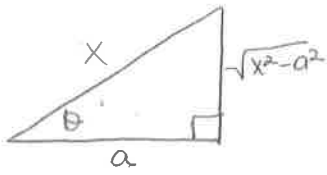
$\boxed{\frac{1}{\sqrt{5}}\sin^{-1}(x\sqrt{5/2}) + C}$

17) $\int \frac{dx}{x\sqrt{x^2-a^2}}$

$x^2 = a^2\sec^2\theta$

$x = a\sec\theta$

$dx = a\sec\theta\tan\theta\, d\theta$



$\sec\theta = \frac{x}{a}$

$x^2 = a^2 + b^2$

$\sqrt{x^2-a^2} = b$

$\theta = \sec^{-1}\frac{x}{a}$

$\int \frac{a\sec\theta\tan\theta\, d\theta}{a\sec\theta\, a\tan\theta}$

Replace

$\sqrt{a^2\sec^2\theta - a^2}$

$\sqrt{a^2(\sec^2\theta - 1)}$

$\sqrt{a^2\tan^2\theta}$

$a\tan\theta$

$\frac{1}{a}\int d\theta$

$\frac{1}{a}\theta + C$

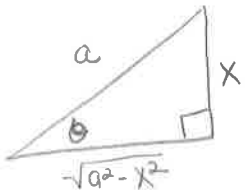
$\boxed{\frac{1}{|a|}\sec^{-1}\left|\frac{x}{a}\right| + C}$

19) $\int \frac{dx}{(a^2-x^2)^{3/2}}$

$x^2 = a^2\sin^2\theta$

$x = a\sin\theta$

$dx = a\cos\theta\, d\theta$



$\sin\theta = \frac{x}{a}$

$a^2 = x^2 + b^2$

$\sqrt{a^2-x^2} = b$

$\int \frac{a\cos\theta\, d\theta}{(a^2\cos^2\theta)^{3/2}}$

Replace

$(a^2 - a^2\sin^2\theta)$

$a^2(1-\sin^2\theta)$

$a^2\cos^2\theta$

$\int \frac{a\cos\theta\, d\theta}{a^3\cos^3\theta}$

$\frac{1}{a^2}\int \frac{1}{\cos^2\theta}\, d\theta$

$\frac{1}{a^2}\int \sec^2\theta\, d\theta$

$\frac{1}{a^2}\tan\theta + C$

$\boxed{\frac{1}{a^2}\cdot\frac{x}{\sqrt{a^2-x^2}} + C}$