VARIATION OF PARAMETERS

NEED TO SOLVE: $y'' + py' + qy = r(x)$

WE KNOW: $y_c = c_1 y_1(x) + c_2 y_2(x)$ IS SOLUTION FROM COMPLEMENTARY EQ.

IT CAN BE SHOWN (PROOF OMITTED) THAT $\boxed{y_p = u(x) y_1(x) + v(x) y_2(x)}$

WE NEED TO FIND $u(x)$ AND $v(x)$.

LET $y_p = u y_1 + v y_2$

$$y_p' = u y_1' + \cancel{y_1 u'} + v y_2' + \cancel{y_2 v'}$$

ASSUME: $\boxed{u' y_1 + v' y_2 = 0}$

$$y_p'' = u y_1'' + y_1' u' + v y_2'' + y_2' v'$$

PLUG ALL OF THESE INTO ORIGINAL DIFF. EQ.

$$y'' + py' + qy = r(x)$$

$$u y_1'' + u' y_1' + v y_2'' + v' y_2' + p(u y_1' + v y_2') + q(u y_1 + v y_2) = r(x)$$

$$u\left(\underbrace{\boxed{y_1'' + p y_1' + q y_1}}_{=0}\right) + v\left(\underbrace{\boxed{y_2'' + p y_2' + q y_2}}_{=0}\right) + \boxed{u' y_1' + v' y_2' = r(x)}$$

$$\therefore \boxed{\begin{aligned} u' y_1 + v' y_2 &= 0 \\ u' y_1' + v' y_2' &= r(x) \end{aligned}}$$

THIS IS A SYSTEM OF EQUATIONS.

WE CAN SOLVE THIS SYSTEM FOR u' AND v'. $\Rightarrow$ FIND u, v.

Ex: $y'' + y = \sec x$

$m^2 + 1 = 0$

$m = 0 \pm i$

$$\boxed{y_c = \cancel{e^{0x}}(c_1 \underbrace{\cos x}_{y_1} + c_2 \underbrace{\sin x}_{y_2})}$$

$$u'(\cos x) + v'(\sin x) = 0$$

$$u'(-\sin x) + v'(\cos x) = \sec x$$

$$u' = \frac{\begin{vmatrix} 0 & \sin x \\ \sec x & \cos x \end{vmatrix}}{\begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}} = \frac{-\tan x}{1} \Rightarrow u = \int -\tan x \, dx$$

$$u = +\ln|\cos x|$$

$$v' = \frac{\begin{vmatrix} \cos x & 0 \\ -\sin x & \sec x \end{vmatrix}}{1} = \frac{1}{1} \Rightarrow v = x$$

$$\boxed{y_p = \ln|\cos x| \cdot \cos x + x \sin x}$$

$$\boxed{y_g = y_c + y_p}$$

$$y'' + 5y' + 6y = e^{-x}$$

$$m^2 + 5m + 6 = 0$$

$$(m+2)(m+3) = 0$$

$$\boxed{y_c = c_1 \underbrace{e^{-2x}}_{y_1} + c_2 \underbrace{e^{-3x}}_{y_2}}$$

$$u' e^{-2x} + v' e^{-3x} = 0$$

$$u'(-2e^{-2x}) + v'(-3e^{-3x}) = e^{-x}$$

$$u' = \frac{\begin{vmatrix} 0 & e^{-3x} \\ e^{-x} & -3e^{-3x} \end{vmatrix}}{\begin{vmatrix} e^{-2x} & e^{-3x} \\ -2e^{-2x} & -3e^{-3x} \end{vmatrix}} = \frac{-e^{-4x}}{-3e^{-5x} + 2e^{-5x}} = \frac{-e^{-4x}}{-e^{-5x}} = e^{x} \Rightarrow u = e^{x}$$

$$v' = \frac{\begin{vmatrix} e^{-2x} & 0 \\ -2e^{-2x} & e^{-x} \end{vmatrix}}{-e^{-5x}} = \frac{e^{-3x}}{-e^{-5x}} = -e^{2x} \Rightarrow v = -\frac{1}{2} e^{2x}$$

$$y_p = e^{x}(e^{-2x}) + -\frac{1}{2} e^{2x}(e^{-3x})$$

$$= e^{-x} - \frac{1}{2} e^{-x} = \boxed{\frac{1}{2} e^{-x}} = y_p$$

$$\boxed{\boxed{y_g = y_c + y_p}}$$