Second Order Differential Equations

Anton 14.1

Second Order Differential Equations:

Equations where the highest derivative is y''

$$y'' + p(x)y' + q(x)y = r(x)$$

Homogeneous: $r(x) = 0$

$$\therefore \quad y'' + p(x)y' + q(x)y = 0$$

Nonhomogeneous: $r(x) \neq 0$

Linear Independence

Two functions are linearly ***dependent*** if one is a constant multiple of the other.

Examples:

1. $f(x) = \sin x$, $\underbrace{g(x) = -4\sin x}_{\text{LINEARLY DEPENDENT}}$

2. $f(x) = \sin x$, $\underbrace{g(x) = x\sin x}_{\text{LINEARLY INDEPENDENT}}$

Theorem

If $y_1(x)$ and $y_2(x)$ are linearly independent solutions of

$$y'' + p(x)y' + q(x)y = 0$$

then $$y_c = c_1 y_1(x) + c_2 y_2(x) \Big\} \text{ LINEAR COMBINATION OF } y_1 \text{ AND } y_2$$

is the general solution of the diff. eq.

y_c is a **linear combination** of y_1 and y_2

Constant Coefficients: $y'' + py' + qy = 0$

What could we try as a solution?

$y = e^{mx}$

$y' = me^{mx}$

$y'' = m^2 e^{mx}$

$m^2 e^{mx} + p \cdot m e^{mx} + q e^{mx} = 0$

$e^{mx}\underbrace{(m^2 + pm + q)}_{0} = 0$

$\boxed{m^2 + pm + q = 0}$ AUXILIARY EQ. CHARACTERISTIC EQ.

SOLVE FOR m:

$m = \dfrac{-p \pm \sqrt{p^2 - 4q}}{2}$

Two Distinct Real Roots: m_1 and m_2

$\therefore\ y_1 = e^{m_1 x}$ and $y_2 = e^{m_2 x}$ are solutions.

The general solution to the differential equation is a linear combination of y_1 and y_2:

$$\boxed{\therefore\ y_c = c_1 e^{m_1 x} + c_2 e^{m_2 x}}$$

Example: $y'' - y' - 6y = 0$

$m^2 - m - 6 = 0$

$(m+2)(m-3)$

$m = -2, 3$

$y_1 = e^{3x}$

$y_2 = e^{-2x}$

$\boxed{y_c = c_1 e^{3x} + c_2 e^{-2x}}$

One Double Root: $m_1 = m_2 = m$

$\therefore\ y_1 = e^{mx}$ and $y_2 = xe^{mx}$ are solutions.

The general solution to the differential equation is a linear combination of y_1 and y_2:

$$\boxed{\therefore\ y_c = c_1 e^{mx} + c_2 x e^{mx}}$$

Example: $y'' - 8y' + 16y = 0$

$m^2 - 8m + 16 = 0$

$(m-4)^2 = 0$

$m = 4$

$\boxed{y_c = c_1 e^{4x} + c_2 x e^{4x}}$

SHOW: $y = xe^{4x}$ IS A SOLUTION

$y' = xe^{4x} \cdot 4 + e^{4x} = e^{4x}(4x+1)$

$y'' = e^{4x}(4) + (4x+1) \cdot 4e^{4x} = e^{4x}(4 + 4(4x+1))$

$= e^{4x}(16x+8)$

$e^{4x}(16x+8) - 8 \cdot e^{4x}(4x+1) + 16xe^{4x} \stackrel{?}{=} 0$

$16x + 8 - 8(4x+1) + 16x \stackrel{?}{=} 0$

✓

Complex Roots: $m_1 = a + bi$, $m_2 = a - bi$

$\therefore\ y_1 = e^{(a+bi)x}$ and $y_2 = e^{(a-bi)x}$ are solutions.

$e^{ax + bix}$

$e^{ax} \cdot e^{bix}$

An equivalent version of y_c that is more commonly used is:

$$\boxed{y_c = e^{ax}\left(c_1 \cos(bx) + c_2 \sin(bx)\right)}$$

Example: $y'' + y' + y = 0$

$m^2 + m + 1 = 0$

$m = \dfrac{-1 \pm \sqrt{1 - 4(1)(1)}}{2}$

$= \dfrac{-1 \pm i\sqrt{3}}{2}$

$= -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$

$\downarrow \quad \downarrow$

$a \quad b$

$y_c = e^{ax}(c_1 \cos bx + c_2 \sin bx)$

$\boxed{= e^{-\frac{1}{2}x}\left(c_1 \cos\left(\frac{\sqrt{3}}{2}x\right) + c_2 \sin\left(\frac{\sqrt{3}}{2}x\right)\right)}$

Example: $y'' - y = 0 \quad y(0) = 1, \quad y'(0) = 0$

$m^2 - 1 = 0$

$m = \pm 1$

$y_c = c_1 e^{x} + c_2 e^{-x} \Rightarrow 1 = c_1 e^{0x} + c_2 e^{0x} \Rightarrow 1 = c_1 + c_2$

$y_c' = c_1 e^{x} - c_2 e^{-x} \Rightarrow 0 = c_1 e^{0x} - c_2 e^{0x} \Rightarrow 0 = c_1 - c_2$

$1 = 2c_1$

$c_1 = 1/2$

$c_2 = 1/2$

$\boxed{y = \frac{1}{2}(e^{x} + e^{-x})} = \cosh(x)$

Classwork:

Anton 14.1 # 10, 18

Homework:

Anton 14.1 # 1 – 23 odd