**AP Calculus BC**
**Chapter 9 Test 2 Review**

1. Let $f(x)=\sum_{k=1}^{\infty}\frac{x^k}{k(k+1)}$. Find the interval and radius of convergence.

2. Consider the series for $f$ defined in Exercise 1 above. Find the interval of convergence for $f'(x)$.

3. Obtain the Taylor series expansion about $x = 0$ for $f(x)=\frac{1}{(1-x)^2}$.

4. The Taylor series for $f(x)=\ln x$ about $x = 1$ may be written in the form $\sum_{n=1}^{\infty}c_n(x-1)^n$. Find a formula for $c_n$.

5. Derive the Taylor series expansion about $x = 0$ for $f(x)=\sin x$ and give the interval of convergence.

6. Find the interval and radius of convergence for $\sum_{k=1}^{\infty}\frac{x^k}{k!}$.

7. Find the interval and radius of convergence for $\sum_{k=1}^{\infty}\frac{(-1)^{k-1}x^{k-1}}{k+1}$.

8. Find the Taylor series expansion about $x = 0$ for $f(x)=3^x$.

9. Given the series $A=\sum_{n=1}^{\infty}\frac{4n}{n^2+1}$

   a. Determine whether the series $A$ converges or diverges. Justify your answer.

   b. If $S$ is the series formed by multiplying the $n$th term in $A$ by the $n$th term in $\sum_{n=1}^{\infty}\frac{1}{2n}$, write an expression using summation notation for $S$.

   c. Determine whether the series $S$ found in part (b) converges or diverges. Justify your answer.

10. Let $f$ be the function defined by $f(x)=\frac{1}{1-2x}$.

    a. Write the first four terms and the general term of the Taylor series expansion about $x = 0$.

    b. What is the interval of convergence for the series found in part (a)? Show your method.

    c. Find the value of $f$ at $x = -\frac{1}{4}$. How many terms of the series are adequate for approximating $f(-1/4)$ with an error not exceeding 1%? Justify your answer.

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11. Let $S$ be the series $\sum_{n=0}^{\infty}\left(\frac{t}{1+t}\right)^n$ where $t \neq 0$.

a. Find the value to which $S$ converges when $t = 1$.

b. Determine the values of $t$ for which $S$ converges. Justify your answer.

c. Find all values of $t$ the make the sum of the series greater than 10.

12. a. Write the Taylor series expansion about $x = 0$ for $f(x) = \ln(1+x)$. Include an expression for the general term.

b. For what values of $x$ does the series in part (a) converge?

c. Estimate the error in evaluating $\ln(3/2)$ by using only the first five nonzero terms of the series in part (a).

d. Use the result in part (a) to determine the logarithmic function whose Taylor series is $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}x^{2n}}{2n}$

13. a. Find the first four nonzero terms in the Taylor series expansion about $x = 0$ for $f(x) = \sqrt{1+x}$.

b. Use the results found in part (a) to find the first four nonzero terms in the Taylor series expansion about $x = 0$ for $g(x) = \sqrt{1+x^3}$.

c. Find the first four nonzero terms in the Taylor series expansion about $x = 0$ for the function $h$ such that $h'(x) = \sqrt{1+x^3}$ and $h(0) = 4$.

# CHAPTER 9 TEST 2 REVIEW

(1) $\rho = \lim_{k\to\infty}\left|\frac{x^{k+1}}{(k+1)(k+2)}\cdot\frac{k(k+1)}{x^k}\right| = |x| < 1$

$x=-1: \sum \frac{(-1)^k}{k(k+1)} \to \text{CONV.}$

$x=1: \sum \frac{1}{k(k+1)} \to \text{CONV.}$

$[-1,1]$, $R=1$

(2) $f' = \sum \frac{\cancel{k}x^{k-1}}{\cancel{k}(k+1)}$

$\rho = \lim_{k\to\infty}\left|\frac{x^k}{k+2}\cdot\frac{k+1}{x^{k-1}}\right| = |x| < 1$

$x=-1: \sum \frac{(-1)^{k-1}}{k+1} \to \text{CONV.}$

$x=1: \sum \frac{1}{k+1} \to \text{DIV}$

$[-1,1)$, $R=1$

(3) $\frac{1}{1-x} = 1+x+x^2+x^3+\cdots+x^n+\cdots$

$\frac{d}{dx}(1-x)^{-1} = \frac{1}{(1-x)^2} = 1+2x+3x^2+\cdots nx^{n-1}+\cdots$

(4) $f=\ln x \quad f(1)=1$

$f' = 1/x \quad f'(1)=1$

$f'' = -x^{-2} \quad f''(1)=-1$

$f''' = 2x^{-3} \quad f'''(1)=2$

$f^{(4)} = -6x^{-4} \quad f^{(4)}(1)=-6$

$= (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \cdots$

$c_n = \frac{(-1)^{k+1}}{k}$

(5) $f=\sin x|_{x=0}=0$

$f' = \cos x|_{x=0} = 1$

$f'' = \sin x|_{x=0} = 0$

$f''' = -\cos x|_{x=0} = -1$

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{(-1)^n x^{2n+1}}{(2n+1)!}$

$\rho = \lim_{n\to\infty}\left|\frac{x^{2n+3}}{(2n+3)!}\cdot\frac{(2n+1)!}{x^{2n+1}}\right| \to 0 \Rightarrow (-\infty,\infty)$

(6) $\rho = \lim_{k\to\infty}\left|\frac{x^{k+1}}{(k+1)!}\cdot\frac{k!}{x^k}\right| = 0 \Rightarrow (-\infty,\infty)$

(7) $\rho = \lim_{k\to\infty}\left|\frac{x^k}{k+2}\cdot\frac{k+1}{x^{k-1}}\right| = |x| < 1$

$x=-1: \sum \frac{(-1)^{k-1}(-1)^{k-1}}{k+1} = \frac{1}{k+1} \to \text{DIV}$

$x=1: \sum \frac{(-1)^{k-1}}{k+1} \to \text{CONV.}$

$(-1,1]$

(8) $f=3^x|_{x=0}=1$

$f' = 3^x\ln 3|_{x=0} = \ln 3$

$f'' = 3^x(\ln 3)^2|_{x=0} = (\ln 3)^2$

$f(x) = 1 + x\ln 3 + \frac{x^2(\ln 3)^2}{2!} + \cdots \frac{x^n(\ln 3)^n}{n!} + \cdots$

(9) a) DIVERGES — USE INTEGRAL TEST

b) $\sum \frac{4n}{n^2+1}\cdot\frac{1}{2n} = \sum \frac{2}{n^2+1}$

c) CONVERGES — USE INTEGRAL TEST

(10) $f = \frac{1}{1-2x}$

a) $\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \cdots$

$\frac{1}{1-2x} = 1 + 2x + (2x)^2 + \cdots (2x)^n + \cdots$

b) $-1 < 2x < 1 \Rightarrow -\frac{1}{2} < x < 1/2$

c) $f(-1/4) = \frac{1}{1-2(-1/4)} = \frac{2}{3}$

$|\text{ERROR}| < .01$

$\left|(2 \cdot -1/4)^{n+1}\right| < .01$

$\left|(-1/2)^{n+1}\right| < \frac{1}{100}$

$n+1 = 7$

$n = 6$

(11) a) $\sum (1/2)^n = 1 + 1/2 + (1/2)^2 + \cdots = \frac{1}{1-1/2} = 2$

b) $\left|\frac{t}{1+t}\right| < 1$

$\left|\frac{1+t}{t}\right| > 1$

$|1/t + 1| > 1$

$1/t + 1 > 1$ OR $1/t + 1 < -1$

$1/t > 0$ $\quad$ $1/t < -2$

$t > 0$ OR $\boxed{t > -1/2}$

c) $a = 1,\ r = t/1+t$

$\text{sum} = \frac{1}{1 - t/1+t} > 10$

$\frac{1+t}{1} > 10$

$t > 9$

(12) a) $\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots (-1)^n x^n + \cdots$

$\ln(1+x) = \int \frac{1}{1+x}\,dx = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}$

$+ (-1)^n \frac{x^{n+1}}{n+1}$

b) $\rho = \lim_{n\to\infty} \left| \frac{x^{n+2}}{n+2} \cdot \frac{n+1}{x^{n+1}} \right| = |x| < 1$

$x = -1: \sum \frac{(-1)^n (-1)^{n+1}}{n+1} = \frac{1}{n+1} \rightarrow \text{DIV}$

$x = 1: \sum \frac{(-1)^n}{n+1} \rightarrow \text{conv} \quad \therefore (-1, 1]$

c) $\sum \frac{(-1)^{n+1} x^{2n}}{2n} = \frac{x^2}{2} - \frac{x^4}{4} + \frac{x^6}{6} - \frac{x^8}{8}$

$= \frac{1}{2} \ln(1 + x^2)$

(13) $f = (1+x)^{1/2}\Big|_{x=0} = 1$

a) $f' = \frac{1}{2}(1+x)^{-1/2}\Big|_{x=0} = \frac{1}{2}$

$f'' = \frac{-1}{4}(1+x)^{-3/2}\Big|_{x=0} = \frac{-1}{4}$

$f''' = \frac{3}{8}(1+x)^{-5/2}\Big|_{x=0} = \frac{3}{8}$

$\therefore f(x) = 1 + \frac{1}{2}x - \frac{1}{4 \cdot 2!}x^2 + \frac{3}{8 \cdot 3!}x^3$

b) $g = 1 + \frac{x^3}{2} - \frac{x^6}{4 \cdot 2!} + \frac{3x^9}{8 \cdot 3!}$

c) $h = \int h'(x)\,dx = x + \frac{x^4}{4 \cdot 2} - \frac{x^7}{7 \cdot 4 \cdot 2!} \cdots + C$

$= 4 + x + \frac{x^4}{4 \cdot 2} - \frac{x^7}{7 \cdot 4 \cdot 2!}$