

# Taylor's Theorem with Lagrange Error

Anton 11.10

Let  $p_n(x)$  denote the  $n$ th Taylor polynomial about  $x = a$  for a function,  $f$ . That means:

$$f(x) = p_n(x) + \text{error}$$

The error component is denoted by  $R_n(x)$  and is called the  $n$ th remainder for  $f$  about  $x = a$ . Therefore,

$$f(x) = p_n(x) + R_n(x)$$

$$|f(x) - p_n(x)| \leq R_n(x)$$



Example: Estimate  $\cos(1)$  using the 4<sup>th</sup> order Maclaurin polynomial for  $\cos(x)$ .

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

$$\cos 1 \approx 1 - \frac{1}{2!} + \frac{1}{4!}$$

Give an upper bound on the error of this estimate.

$$|\text{ERROR}| \leq \left| \frac{x^6}{6!} \right|_{x=1} = \frac{1}{6!}$$



If the series is alternating and converges, we know the error of an estimate will always be less than or equal to the magnitude of the next unused term.

What if the series is not alternating? We need another way to find an upper bound on the error.



**Remainder Estimation Theorem**  
(Lagrange form of the remainder)

$$\left| R_n(x) \right| \leq \left| \frac{M}{(n+1)!} (x-a)^{n+1} \right|$$

where  $M$  is the maximum value of  $f^{(n+1)}(x)$



**Example:** The function  $f$  has derivatives of all orders for all real numbers  $x$ . Assume the following:

$$f(-3) = -2, f'(-3) = 4, f''(-3) = -6, f'''(-3) = 12$$

- a. Write the third-degree Taylor polynomial about  $x = -3$  and use it to approximate  $f(-3.5)$ .

$$p_3 = f(-3) + f'(-3)(x+3) + \frac{f''(-3)}{2!}(x+3)^2 + \frac{f'''(-3)}{3!}(x+3)^3$$

$$f(x) \approx p_3 = -2 + 4(x+3) - \frac{6}{2!}(x+3)^2 + \frac{12}{3!}(x+3)^3$$

$$f(-3.5) \approx p_3(-3.5) = -2 + 4(-1/2) - 3(-1/2)^2 + 2(-1/2)^3 = \underline{\underline{-5}}$$



**Example:** The function  $f$  has derivatives of all orders for all real numbers  $x$ . Assume the following:

$$f(-3) = -2, f'(-3) = 4, f''(-3) = -6, f'''(-3) = 12$$

- b. The fourth derivative of  $f$  satisfies the inequality below for all  $x$  in the closed interval  $[-3.5, -3]$ . Use the Lagrange error bound on the estimate found in part a) to explain why  $f(-3.5) > -5.15$ .

$$|f^{(4)}(x)| \leq 48$$

$$|R_3(x)| \leq \left| \frac{M}{(n+1)!} \cdot (x-a)^{n+1} \right| = \left| \frac{48}{4!} (-1/2)^4 \right| = 1/8$$

$f(-3.5)$  IS A MAXIMUM WITHIN THE INTERVAL  $-5 \pm 1/8$   
OR  $(-5.125, -4.875)$   
SO ACTUAL VALUE MUST BE  $> -5.15$ .



**Example:** The function  $f$  has derivatives of all orders for all real numbers  $x$ . Assume the following:

$$f(-3) = -2, f'(-3) = 4, f''(-3) = -6, f'''(-3) = 12$$

- c. Write the fourth-degree Taylor polynomial for  $g(x) = f(x^2 - 3)$  about  $x = 0$ . Use the polynomial to find  $x$  - values where  $g$  has relative extrema.

Justify your answer.

$$p_3 = -2 + 4(x+3) - 3(x+3)^2$$

$$g(x) \approx f(x^2-3) \approx -2 + 4(x^2) - 3(x^4)$$

$$g' = 8x - 12x^3 = 0 \Rightarrow x = 0$$

$$4x(2 - 3x^2) = 0$$

$$g'' = 8 - 36x^2 \Big|_{x=0} = 8 > 0$$

$\therefore$  rel. min @  $x=0$ .



## AP Packet #35 (2006B #6) parts a, b

$$a) f(x) = 1 - x^3 + x^6 - x^9 + \dots + (-1)^n x^{3n} + \dots$$

$$f'(x) = -3x^2 + 6x^5 - 9x^8 + \dots + (-1)^n \cdot 3n x^{3n-1} + \dots$$

$$b) -\frac{3}{2^2} + \frac{6}{2^6} - \frac{9}{2^9} + \dots \quad \text{NOT GEOMETRIC, BUT IT DOES EQUAL } f'(1/2)$$

$$f(x) = \frac{1}{1+x^3} \Rightarrow f'(x) = -1(1+x^3)^{-2} \cdot 3x^2 = \frac{-3x^2}{(1+x^3)^2} \Big|_{x=1/2}$$

$$= \frac{-3 \cdot \frac{1}{4}}{(1 + \frac{1}{8})^2} = \frac{-3/4}{81/64}$$

$$= \frac{-3 \cdot 64}{4 \cdot 81}$$

$$= \frac{-16}{27}$$



## AP Packet #35 (2006B #6) - parts c, d

$$c) \int_0^{\infty} f(x) dx = x - \frac{1}{4}x^4 + \frac{1}{7}x^7 - \frac{1}{10}x^{10} + \dots + (-1)^n \frac{1}{3n+1} x^{3n+1} + \dots$$

$$d) \int_0^{1/2} f(x) dx \approx \left[ \frac{1}{2} - \frac{1}{4} \left(\frac{1}{2}\right)^4 + \frac{1}{7} \left(\frac{1}{2}\right)^7 \right]$$

MAX TERMS DECREASING TO ZERO  $\Rightarrow$  BY ALT SERIES TEST, IT CONV.

$$|\text{error}| \leq \left| \frac{1}{10} \cdot \left(\frac{1}{2}\right)^{10} \right| = \left| \frac{1}{10 \cdot 1024} \right| = \frac{1}{10240} < \frac{1}{10,000}$$



# **Homework:**

## **Chapter 9 AP Packet**

**24. 1999 BC4**

**28. 2003 BC6**

**30. 2004 BC6**

