Alternating Series Error Approximation

Anton 11.7

Recall the alternating harmonic series:

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \qquad = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

ALT SERIES TEST

We know that this series converges (why?), but what do we <u>not</u> know?

Let’s try and estimate the sum.

Estimating the Sum:

Look at partial sums:

$$S_1 = 1$$

$$S_2 = 1 - 1/2 = 0.5$$

$$S_3 = 1 - 1/2 + 1/3 = 0.8333$$

$$S_4 = 1 - 1/2 + 1/3 - 1/4 = 0.5833$$

So far, we can say the actual sum is between what two values?

If we continue:

$$S_7 = 1 - 1/2 + \cdots + 1/7 = 0.7595$$

$$S_8 = 1 - 1/2 + 1/3 - \cdots - 1/8 = 0.6345$$

So far, we can say the actual sum is between what two values?

In general, the actual sum will always be between S_n and S_{n+1} (any consecutive partial sums.)

ASSUMING A CONVERGING ALT. SERIES

What is the error for the seventh partial sum?

$$S_7 = 1 - 1/2 + \cdots + 1/7 = 0.7595$$ $\longrightarrow$ ERROR OF $S_7 \le |1/8|$

$$S_8 = 1 - 1/2 + 1/3 - \cdots - 1/8 = 0.6345$$ $\longrightarrow$ ERROR $\le |1/9|$

"NTH PARTIAL SUM"

In general, if you use S_n to approximate the value of the actual sum, S, then the error will be:

(n+1) TH TERM (NEXT TERM)

ACTUAL VALUE ESTIMATE

$$error = |S - S_n| \le |a_{n+1}|$$

The actual value of the sum is ln(2). Verify that the error of each estimate is less than the appropriate value.

$$S_7 = 0.7595 \Rightarrow error = \left|\ln 2 - .7595\right| = 0.0664 < \left|a_8\right| = 1/8$$

$$S_8 = 0.6345 \Rightarrow error = \left|\ln 2 - .6345\right| = 0.0586 < \left|a_9\right| = 1/9$$

What partial sum should we use if we want to estimate S to within 0.0001?

$|\text{ERROR}| \leq .0001$

In other words, if we use S_n to approximate S to within 0.0001, what value of n should we use?

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$$

$$\left|a_{n+1}\right| \leq 0.0001$$

$$\frac{1}{n+1} \leq 0.0001 = \frac{1}{10{,}000}$$

$$n+1 \geq 10{,}000$$

$n \geq 9{,}999$

Therefore, take $n = 9999$

$|S - S_{9999}| \leq .0001$

Assume we take the smallest n.

Given the series: $$\sum_{k=1}^{\infty} \frac{(-1)^k 2^k}{k!}$$

$$S_n = \frac{-2}{1!} + \frac{2^2}{2!} - \frac{2^3}{3!} + - \frac{(-1)^n 2^n}{n!} \quad \frac{(-1)^{n+1} 2^{n+1}}{(n+1)!}$$

ERROR

We showed yesterday that this series converges.

Find the value of *n* for which the *n*th partial sum will approximate the sum of the series to within 0.0001?

$$|\text{ERROR}| \le .0001 = \frac{1}{10{,}000}$$

$$\frac{2^{n+1}}{(n+1)!} \le \frac{1}{10{,}000}$$

$$\boxed{\frac{(n+1)!}{2^{n+1}} \ge 10{,}000}$$

GUESS + CHECK:

$$\frac{(10+1)!}{2^{10+1}} \approx 19{,}000 \ge 10{,}000$$

$$\boxed{\text{TAKE } n = 10}$$

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$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k!} \qquad n = 5$$

$$S_5 = \boxed{\frac{1}{1!} - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \frac{1}{5!}} - \frac{1}{6!}$$

ERROR

$$\left| \frac{1}{6!} \right|$$

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$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(k+1)\ln(k+1)} \qquad n = 3$$

$$S_3 = \boxed{\frac{1}{2\ln 2} - \frac{1}{3\ln 3} + \frac{1}{4\ln 4}} - \frac{1}{5\ln 5}$$

$$|\text{ERROR}| \leq \frac{1}{5\ln 5}$$

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$$\sum \frac{(-1)^{k+1}}{k!} \qquad |ERROR| < .0001$$

$$|ERROR| < \frac{1}{100{,}000}$$

$$|a_{n+1}| \le \frac{1}{100{,}000}$$

$$\frac{1}{(n+1)!} \le \frac{1}{100{,}000}$$

$$(n+1)! \ge 100{,}000$$

$$(8+1)! \ge 100{,}000$$

$$\boxed{\text{TAKE } n = 8}$$

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$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(k+1)\ln(k+1)} \qquad |ERROR| < 1/10$$

$$|ERROR| < 1/10$$

$$|a_{n+1}| \le 1/10$$

$$\frac{1}{(n+2)\ln(n+2)} \le \frac{1}{10}$$

$$(n+2)\ln(n+2) \ge 10$$

$$6 \ln 6 \ge 10$$

$$\boxed{n = 4}$$

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40 $\quad 1 - 2/3 + 4/9 - 8/27 + \cdots$

GEOM w/ $r = -2/3 \quad |r|<1 \Rightarrow$ CONV

$$\sum_{k=\cancel{0}\,1}^{\infty} \left(-\frac{2}{3}\right)^{k-1}$$

$$|\text{ERROR}| \le |a_{11}| = \boxed{\left|\left(-2/3\right)^{10}\right|}$$

$$= (2/3)^{10}$$

$|\text{ERROR}| < \frac{1}{10{,}000}$

42 $\quad \cos 1 = \boxed{\frac{1}{0!} - \frac{1}{2!} + \frac{1}{4!} - \frac{1}{6!}} + \cdots \frac{1}{8!}$

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(2k-2)!}$$

$$|a_{n+1}| \le \frac{1}{10{,}000}$$

$$\left|\frac{1}{(2(n+1)-2)!}\right| \le \frac{1}{10{,}000}$$

$$(2n)! \ge 10{,}000$$

$$(2\cdot 4)! \ge 10{,}000$$

$\boxed{n=4}$

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Homework:

Anton 11.7 # 31 – 45 odd