Alternating Series

Anton 11.7

Tests for Convergence/Divergence (so far):

1. Geometric Series Test
2. P-series Test (included harmonic series)
3. Divergence Test
4. Integral Test
5. Ratio Test
6. Root Test
7. Limit Comparison Test
8. Comparison Test

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Informal Principle #1

Constant terms in the denominator can be ignored without affecting the convergence or divergence of a series.

$$\sum_{k=1}^{\infty} \frac{1}{2^k \cancel{+1}}$$

BEHAVES LIKE GEOMETRIC

w/ $r = 1/2$

$$\Rightarrow \sum_{k=1}^{\infty} \frac{1}{2^k+1} \quad \underline{\text{CONV}}$$

$$\sum_{k=1}^{\infty} \frac{1}{\sqrt{k} \cancel{-2}}$$

BEHAVES LIKE P-SERIES

$p = 1/2$

$$\Rightarrow \sum_{k=1}^{\infty} \frac{1}{\sqrt{k}-2} \quad \underline{\underline{\text{DIV}}}$$

$$\sum_{k=1}^{\infty} \frac{1}{\left(k + \cancel{\frac{1}{2}}\right)^3}$$

BEHAVES LIKE P-SERIES, $p = 3$

$$\sum_{k=1}^{\infty} \frac{1}{(k+1/2)^3} \quad \underline{\text{CONV}}$$

Informal Principle #2

Highest powers of *k* matter the most in a polynomial. Ignoring the rest will not affect the convergence or divergence of the series.

$$\sum_{k=1}^{\infty} \frac{1}{\sqrt{k^3 + \cancel{2k}}}$$

BEHAVE LIKE P-SERIES, $p = 3/2$

$\sum \frac{1}{\sqrt{k^3+2k}}$ CONV.

"BEHAVES LIKE"

$$\sum_{k=1}^{\infty} \frac{6k^4 - \cancel{2k^3} + \cancel{1}}{k^5 + \cancel{k^2} - \cancel{2k}} \sim \sum_{k=1}^{\infty} \frac{6}{k}$$

BEHAVES LIKE A HARMONIC $\Rightarrow$ DIVERGES.

Alternating Series

$a_k > 0$

In general, an alternating series has one of the following two forms:

$$\sum_{k=1}^{\infty} (-1)^{k+1} a_k = a_1 - a_2 + a_3 - a_4 + \cdots$$

$$\sum_{k=1}^{\infty} (-1)^{k} a_k = -a_1 + a_2 - a_3 + a_4 - \cdots$$

In both cases, assume a_k is positive.

Alternating Series Test

An alternating series converges if the following are both true:

1. $a_1 > a_2 > a_3 > \cdots$
2. $\lim\limits_{k \to \infty} a_k = 0$

MAGNITUDE

The absolute value of the terms (disregard sign) are decreasing to 0.

Ex: $$\sum_{k=1}^{\infty}\frac{(-1)^k}{k}$$

ALTERNATING HARMONIC

$$= -1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \cdots \quad \cdot\cdot$$

THE MAGNITUDE OF TERMS ARE DECREASING TO 0 $\Rightarrow$ CONVERGES.

$$\boxed{\sum_{k=1}^{\infty}\frac{(-1)^k}{k} \quad \underline{\underline{\text{CONV}}}}$$

Ex: $$\sum_{k=1}^{\infty}\frac{(-1)^{k+1}(k+3)}{k(k+1)}$$

$$= \frac{4}{1(2)} - \frac{5}{2(3)} + \frac{6}{3(4)} - \frac{7}{(4)(5)} + \cdots$$

THE MAG. OF TERMS ARE DECREASING TO 0

$$\sum_{k=1}^{\infty}\frac{(-1)^{k+1}(k+3)}{(k)(k+1)} \quad \underline{\underline{\text{CONV}}}.$$

Absolute Convergence:

An alternating series will converge *absolutely* if:

$$\sum_{k=1}^{\infty}\left|(-1)^k a_k\right| = a_1 + a_2 + a_3 + \cdots \text{ converges.}$$

$$\sum_{k=1}^{\infty}\frac{(-1)^k}{2^k} \rightarrow \text{CONV ABSOLUTELY}$$

$$\sum\left|\frac{(-1)^k}{2^k}\right| = \sum\frac{1}{2^k}$$ GEOM w/ r = 1/2 $\Rightarrow$ CONV

$$\sum_{k=1}^{\infty}\frac{(-1)^k}{k} \rightarrow \text{DOES NOT CONV ABSOLUTELY}$$

$$\sum\left|\frac{(-1)^k}{k}\right| = \sum\frac{1}{k} \rightarrow \text{DIVERGES.}$$

Theorem: If a series converges absolutely, then it converges (two for one.)

Conditional Convergence:

If an alternating series converges but the series of absolute values does not, then the original alternating series converges *conditionally*.

Ex: $\sum_{k=1}^{\infty} \frac{(-1)^k}{k}$ CONV CONDITIONALLY

Ratio Test for Absolute Convergence

Let Σu_k have nonzero terms.

Let $\rho = \lim_{k\to\infty} \left| \frac{u_{k+1}}{u_k} \right|$ $= \lim_{k\to\infty} \left| u_{k+1} \cdot \frac{1}{u_k} \right|$

1. If $\rho < 1$ then the series converges *absolutely*.
2. If $\rho > 1$ then the series diverges.
3. If $\rho = 1$ then the test is inconclusive*.

*Further examination is required.

Ex: $\sum_{k=1}^{\infty}(-1)^k \frac{2^k}{k!}$

$$\rho = \lim_{k\to\infty}\left|\frac{u_{k+1}}{u_k}\right|$$

$$= \lim_{k\to\infty}\left|\frac{2^{\cancel{k+1}}}{(k+1)\cancel{!}}\cdot\frac{\cancel{k!}}{\cancel{2^k}}\right| = 0 < 1 \quad \therefore \sum (-1)^k \frac{2^k}{k!} \text{ conv } \underline{\underline{\text{ABSOLUTELY}}}$$

$$\left(\sum \frac{2^k}{k!}\right)$$

Tests for Convergence/Divergence (so far):

1. Geometric Series Test   conv $|r|<1$
2. P-series Test (including harmonic)   conv $p>1$
3. Divergence Test
4. Integral Test   conv ABSOLUTELY
5. Ratio Test for Absolute Convergence $\rho<1$
6. Root Test (for Absolute Convergence)

~~7.~~ Limit Comparison Test

~~8.~~ Comparison Test

9. Alternating Series Test

Homework:

Anton 11.7 # 1 – 29 odd