Additional Convergence Tests

Anton 11.5

Tests for Convergence/Divergence (so far):

1. Geometric Series Test
2. P-series Test (included harmonic series)
3. Divergence Test
4. Integral Test
5. Special Cases – telescoping series

Ratio Test

Let Σu_k have positive terms.

Let $\boxed{\rho = \lim_{k\to\infty} \frac{u_{k-1}}{u_k}}$ $\quad u_{k+1} \cdot \frac{1}{u_k}$

1. If $\rho < 1$ then the series converges.
2. If $\rho > 1$ then the series diverges.
3. If $\rho = 1$ then the test in inconclusive*.

*Further examination is required.

Ex: $\sum_{k=1}^{\infty} \underbrace{\frac{1}{k!}}_{u_k}$

$$\rho = \lim_{k\to\infty} \frac{1}{(k+1)\cancel{k!}} \cdot \frac{\cancel{k!}}{1} \to 0 < 1$$

$\therefore \sum \frac{1}{k!}$ CONVERGES.

Ex: $\sum_{k=1}^{\infty} \frac{k}{2^k}$

$$\rho = \lim_{k\to\infty} \frac{k+1}{2^{\cancel{k+1}}} \cdot \frac{\cancel{2^k}}{\cancel{k}} = \frac{1}{2} < 1$$

$\therefore \sum \frac{k}{2^k}$ CONVERGES.

Ex: $\sum_{k=1}^{\infty} \frac{k^k}{k!}$

$$\rho = \lim_{k\to\infty} \frac{(k+1)^{\cancel{k+1}\,k}}{\cancel{(k+1)!}} \cdot \frac{\cancel{k!}}{k^k} = \frac{(k+1)^k}{k^k} = \left(\frac{k+1}{k}\right)^k$$

$$y = \left(1+\frac{1}{k}\right)^k$$

$$\ln y = k \ln\left(1+\frac{1}{k}\right) = \frac{\ln(1+1/k)}{1/k} \xrightarrow{L} \frac{\frac{1}{1+1/k}\cdot \frac{-1}{k^2}}{-1/k^2} \to 1$$

$\ln y \to 1 \Rightarrow \boxed{y \to e}$ $\therefore \rho = e > 1 \therefore \boxed{\sum \frac{k^k}{k!} \text{ DIVERGES}}$

Ex: $\sum_{k=1}^{\infty} \frac{(2k)!}{4^k}$

$$\rho = \lim_{k\to\infty} \frac{(2(k+1))!}{4^{k+1}} \cdot \frac{4^k}{(2k)!} = \frac{\overset{(2k+2)(2k+1)\cancel{(2k)}}{(2k+2)!}}{4^{\cancel{k+1}}} \cdot \frac{\cancel{4^k}}{\cancel{(2k)!}} = \frac{(2k+2)(2k+1)}{4}$$

$\to \infty$

$\therefore \rho > 1$

$\therefore \sum \frac{(2k)!}{4^k}$ DIVERGES

Ex: $\sum_{k=1}^{\infty} \frac{1}{2k-1}$

$$\rho = \lim_{k\to\infty} \frac{1}{2(k+1)-1} \cdot \frac{2k-1}{1} = \frac{2k-1}{2k+1} \to 1$$

$\therefore$ RATIO TEST IS INCONCLUSIVE.

Root Test

Let Σu_k have positive terms.

Let $\rho = \lim_{k \to \infty} \sqrt[k]{u_k} = \lim_{k \to \infty} (u_k)^{\frac{1}{k}}$

1. If $\rho < 1$ then the series converges.
2. If $\rho > 1$ then the series diverges.
3. If $\rho = 1$ then the test in inconclusive*.

*Further examination is required.

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Examples:

$$\sum_{k=1}^{\infty} \left(\frac{4k-5}{2k+1} \right)^k$$

$$\rho = \lim_{k \to \infty} \left[\left(\frac{4k-5}{2k+1} \right)^k \right]^{1/k} = \frac{4k-5}{2k+1} \longrightarrow 2 > 1$$

$\therefore \sum \left(\frac{4k-5}{2k+1} \right)^k$ DIVERGES

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Examples:

$$\sum_{k=1}^{\infty} \frac{1}{(\ln(k+1))^k}$$

$$\rho = \lim_{k \to \infty} \left[\frac{1}{(\ln(k+1))^k} \right]^{1/k} = \frac{1}{\ln(k+1)} \longrightarrow 0 < 1$$

$\therefore \sum \frac{1}{(\ln(k+1))^k}$ CONVERGES

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Tests for Convergence/Divergence (so far):

1. Geometric Series Test
2. P-series Test (included harmonic series)
3. Divergence Test
4. Integral Test
5. Ratio Test
6. Root Test

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Homework:

Anton 11.5 # 1 – 25 odd