Convergence Tests

Anton 11.4

Types of series we've discussed so far:

1. Geometric
2. Telescoping
3. Harmonic

We need some more tests to determine whether a given series converges.

<u>Divergence Test (nth term test)</u>

Let a_k be the general term of Σa_k.

If $\boxed{\lim_{k\to\infty} a_k \neq 0 \text{ then } \Sigma a_k \text{ diverges.}}$

If $\lim_{k\to\infty} a_k = 0$ then Σa_k may converge or diverge.

Thrm: If Σa_k converges then $\lim(a_k) = 0$

Is the converse of this theorem true?

Some examples:

$$\sum_{k=1}^{\infty} \frac{k}{k+1}$$

$\lim_{k\to\infty} \frac{k}{k+1} = 1 \Rightarrow \therefore \sum \frac{k}{k+1}$ DIV

$$\sum_{k=1}^{\infty} \frac{1}{k}$$

$\lim_{k\to\infty} \frac{1}{k} = 0$ THE DIVERGENCE TEST IS INCONCLUSIVE. $\sum 1/k$ DIVERGES (HARMONIC)

$$\sum_{k=1}^{\infty} \left(\frac{1}{2}\right)^k$$

$\lim_{k\to\infty} \left(\frac{1}{2}\right)^k = 0$ DIVERGENCE TEST IS INCONCLUSIVE. GEOMETRIC $r = 1/2$ $\therefore$ CONVERGES.

Properties of Series:

1. If Σu_k and Σv_k are convergent series, then $\Sigma(u_k \pm v_k)$ is convergent. Moreover,

$$\boxed{\sum(u_k \pm v_k) = \sum u_k \pm \sum v_k}$$

Ex: $\displaystyle\sum_{k=1}^{\infty}\left(\frac{3}{4^k} - \frac{2}{5^{k-1}}\right) = \sum \frac{3}{4^k} - \sum \frac{2}{5^{k-1}}$

$a = 3/4$, $r = 1/4$; $a = 2$, $r = 1/5$

$$\left(\frac{3/4}{1-1/4} - \frac{2}{1-1/5}\right) = -1.5$$

Properties of Series (cont.):

2. $\Sigma cu_k = c\Sigma u_k$

Ex: $\displaystyle\sum_{k=1}^{\infty}\left(\frac{5}{k}\right) = 5\sum \frac{1}{k} \Rightarrow$ DIVERGES

Properties of Series (cont.):

3. The convergence or divergence of a series is not affected by removing a finite number of terms*.

Ex: $\displaystyle\sum_{k=10}^{\infty}\left(\frac{1}{k}\right) = \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \cdots \rightarrow$ DIVERGES (HARMONIC)

Ex: $\displaystyle\sum_{k=10}^{\infty}\left(\frac{1}{2}\right)^k$ Geom w/ $r = 1/2$ $\therefore$ conv. $\quad$ sum $= \dfrac{(1/2)^{10}}{1-1/2} = \dfrac{(1/2)^{10}}{1/2} = (1/2)^9$

*The sum *is* affected (if the series converges.)

Integral Test

Let Σa_k have positive terms. Let $f(x) = a_x$. If $f(x)$ is decreasing on $[b,\infty)$ then:

$$\sum_{k=1}^{\infty} a_k \quad \text{and} \quad \overset{\text{IMPROPER}}{\int_b^{\infty} f(x)\,dx}$$

both converge or **both** diverge.

Examples:

$$\sum_{k=1}^{\infty}\frac{1}{k}$$

$\int_1^{\infty}\frac{1}{x}\,dx$

$\ln|x|\Big]_1^{\infty}$

$\ln\infty-\ln 1$

∞

$\therefore \int_1^{\infty}\frac{1}{x}\,dx$ DIV.

$\therefore \sum\frac{1}{k}$ DIV.

$$\sum_{k=1}^{\infty}\frac{1}{k^2}$$

$\int_1^{\infty}\frac{1}{x^2}\,dx$

$-\frac{1}{x}\Big]_1^{\infty}$

$-\frac{1}{\infty}+\frac{1}{1}$

$1 \Rightarrow$ THIS IS NOT THE SUM OF SERIES

$\therefore \int_1^{\infty}\frac{1}{x^2}\,dx$ CONV.

$\therefore \sum_{k=1}^{\infty}\frac{1}{k^2}$ CONV.

For what values of p will the following series converge? $(p>0)$

$$\sum_{k=1}^{\infty}\frac{1}{k^p}$$

$\int_1^{\infty}\frac{1}{x^p}\,dx = \int_1^{\infty}x^{-p}\,dx$

$p=1$: $\int_1^{\infty}\frac{1}{x}\,dx \Downarrow$ DIVERGES

$\frac{1}{1-p}x^{-p+1}\Big]_1^{\infty}$

$\frac{1}{1-p}\left(\infty^{\boxed{-p+1}}-1\right)$

THIS CONVERGES IF $-p+1<0$

$1<p$

$p>1$

P-series Test

If $p>0$, then a **p-series** has the following form:

$$\sum_{k=1}^{\infty}\frac{1}{k^p}=1+\frac{1}{2^p}+\frac{1}{3^p}+\frac{1}{4^p}+\cdots$$

A p-series will *converge* if: $p>1$

A p-series will *diverge* if: $0<p\le 1$

Examples:

$\sum_{k=1}^{\infty}\frac{1}{k}$ P-SERIES, $p=1$ DIVERGES

$\sum_{k=1}^{\infty}k^{-\frac{3}{2}}=\frac{1}{k^{3/2}}$ P-SERIES $p=3/2$ CONVERGES

$\sum_{k=1}^{\infty}\frac{1}{k^2}$ P-SERIES, $p=2$ CONVERGES

$1+\frac{1}{\sqrt[3]{2}}+\frac{1}{\sqrt[3]{3}}+\cdots \sum\frac{1}{k^{1/3}}$ $p=1/3$ DIVERGES

$\sum_{k=1}^{\infty}\frac{1}{\sqrt{k}}$ P-SERIES, $p=1/2$ DIVERGES.

Tests for Convergence/Divergence (so far):

1. Geometric Series Test
2. P-series Test (included harmonic series)
3. Divergence Test
4. Integral Test
5. Special Cases – telescoping series

Homework:

Anton 11.4 # 1 – 31 odd