Infinite Series

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Infinite series:

$$\sum_{k=1}^{\infty} a_k = a_1 + a_2 + a_3 + \cdots$$

How is a series different from a sequence?

Ex: $\sum_{k=1}^{\infty} \frac{3}{10^k} = \frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} + \cdots \longrightarrow 1/3$

Does the series converge? To answer, we need to look at the sequence of partial sums:

A sequence of partial sums, $\{S_n\}$, is defined as:

$\{S_n\} = s_1, s_2, s_3, \ldots$ where

$$s_1 = a_1$$
$$s_2 = a_1 + a_2$$
$$s_3 = a_1 + a_2 + a_3$$
$$s_n = a_1 + a_2 + a_3 + \cdots + a_n$$

For our problem:

$$s_1 = \frac{3}{10} = .3$$
$$s_2 = \frac{3}{10} + \frac{3}{100} = .33$$
$$s_3 = \frac{3}{10} + \frac{3}{100} + \frac{3}{1000} = .333$$
$$\therefore \{S_n\} = 0.3, 0.33, 0.333, 0.3333, \ldots \rightarrow 0.333\ldots = \frac{1}{3}$$
$$\boxed{\therefore \sum_{k=1}^{\infty} \frac{3}{10^k} = \frac{1}{3}}$$

More formally:

Let $\{S_n\}$ be a sequence of partial sums of Σa_k.

If $\{S_n\} \to L$ then $\Sigma a_k = L$

If $\{S_n\}$ diverges then Σa_k diverges.

Ex: $\sum_{k=1}^{\infty} (-1)^{k+1} = 1 - 1 + 1 - 1 + 1 - 1 + \cdots$

$\{S_n\} = 1, 0, 1, 0, 1, 0, \cdots \to$ DIVERGES $\Rightarrow$ $\boxed{\Sigma(-1)^{k+1} \text{ DIVERGES}}$

Geometric Series: a series of the form

$$\sum_{k=1}^{\infty} ar^{k-1} = a + ar + \underbrace{ar^2 + ar^3}_{\frac{ar^3}{ar^2} = r} + \cdots \qquad a \neq 0$$

What do you do to each term to get the next term? MULT BY r

r is called the ***common ratio***.

Identify *a* (the first term) and *r* (the common ratio):

$\sum_{k=1}^{\infty} 2^{k-1} = 1 + 2^1 + 2^2 + \cdots$ $a = 1$ $r = 2$	$1 + 1 + 1 + \ldots$ $a = 1$ $r = 1$
$\sum_{k=1}^{\infty} \frac{3}{10^k}$ $a = 3/10$ $r = 1/10$	$1 - 1 + 1 - 1 + \ldots$ $a = 1$ $r = -1$
$\sum_{k=1}^{\infty} \left(-\frac{1}{2}\right)^k = \left(-\frac{1}{2}\right) + \left(-\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right)^3 + \cdots$ $a = -1/2$ $r = -1/2$	

When do you think a geometric series will converge?

Convergence of a Geometric Series:

A geometric series will **converge** if $|r| < 1$.

If the geometric series **converges**, then the sum is:

$$\sum_{k=1}^{\infty} ar^{k-1} = \frac{a}{1-r}$$

A geometric series **diverges** if $|r| \geq 1$

Some other examples:

$$\sum_{k=1}^{\infty}\frac{5}{4^{k-1}} = 5+\frac{5}{4}+\frac{5}{4^2}+\cdots \qquad \text{sum} = \frac{a}{1-r} = \frac{5}{1-1/4} = \frac{5}{3/4} = \boxed{\frac{20}{3}}$$

$a = 5$ CONV. SINCE

$r = 1/4$ $|r|<1$

$$\sum_{k=1}^{\infty}\frac{3}{(-7)^{k+1}} = \frac{3}{(-7)^2}+\frac{3}{(-7)^3}+\cdots \qquad \text{sum} = \frac{3/49}{1+1/7} = \frac{3/49}{8/7} = \frac{3}{\cancel{49}_{7}}\cdot\frac{\cancel{7}}{8} = \boxed{\frac{3}{56}}$$

$a = 3/49$ CONV. SINCE

$r = -1/7$ $|r|<1$

Some other examples:

$$0.\overline{7} = \frac{7}{10}+\frac{7}{100}+\frac{7}{1000}+\cdots = \frac{7/10}{1-1/10} = \frac{7/10}{9/10} = \boxed{7/9}$$

$a = 7/10$ CONV. SINCE

$r = 1/10$ $|r|<1$

$$3.\overline{62} = 3+\underbrace{\frac{62}{100}+\frac{62}{100^2}+\frac{62}{100^3}+\cdots}_{} \qquad \frac{a}{1-r} = \frac{62/100}{1-1/100} = \frac{62}{99}$$

$a = 62/100$

$r = 1/100$

$$\Downarrow$$

$$\boxed{3\frac{62}{99}}$$

Harmonic Series

$$\sum_{k=1}^{\infty}\frac{1}{k} = 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots$$

Does this series converge?

NO!

Telescoping Series – an example

$$\sum_{k=1}^{\infty}\frac{1}{k(k+1)} = \frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots \longrightarrow 1$$

$$\sum \frac{1}{k}+\frac{-1}{k+1} = \left(1-\cancel{1/2}\right)+\left(\cancel{1/2}-\cancel{1/3}\right)+\left(\cancel{1/3}-\cancel{1/4}\right)+\cdots$$

Telescoping Series – the terms will only cancel out if they are getting smaller.

Homework:

Anton 11.3 # 3 – 29 odd