Comparison Tests

Anton 11.6

Tests for Convergence/Divergence (so far):

1. Geometric Series Test
2. P-series Test (included harmonic series)
3. Divergence Test
4. Integral Test
5. Ratio Test
6. Root Test

Informal Principle #1

Constant terms in the denominator can be ignored without affecting the convergence or divergence of a series.

$$\sum_{k=1}^{\infty} \frac{1}{2^k \cancel{+1}}$$

THIS SERIES "BEHAVES" LIKE $\sum \frac{1}{2^k}$ GEOM. $r = 1/2$ $\Rightarrow$ CONV.

$$\sum_{k=1}^{\infty} \frac{1}{\sqrt{k \cancel{-2}}} = \frac{1}{k^{1/2}}$$

THIS SERIES IS LIKE A P-SERIES ($p = 1/2$) $\Rightarrow$ DIVERGES

$$\sum_{k=1}^{\infty} \frac{1}{\left(k + \cancel{\frac{1}{2}}\right)^3}$$

THIS SERIES IS LIKE A P-SERIES ($p=3$) $\Rightarrow$ CONVERGES

Informal Principle #2

Highest powers of k matter the most in a polynomial. Ignoring the rest will not affect the convergence or divergence of the series.

$$\sum_{k=1}^{\infty} \frac{1}{\sqrt{k^3 \cancel{+2k}}} \approx \frac{1}{\sqrt{k^3}} = \frac{1}{k^{3/2}}$$

THIS SERIES IS LIKE A P-SERIES ($p = 3/2$) $\Rightarrow$ CONVERGES

$$\sum_{k=1}^{\infty} \frac{6k^4 \cancel{-2k^3} \cancel{+1}}{k^5 \cancel{+k^2} \cancel{-2k}} \approx \frac{6}{k}$$

THIS SERIES IS LIKE A HARMONIC SERIES $\Rightarrow$ DIVERGES

Limit Comparison Test

Let Σa_k and Σb_k have positive terms.

Let $\rho = \lim_{k\to\infty} \frac{a_k}{b_k}$

If ρ is **finite** and $\rho > 0$ then **both** series converge or both series diverge.

Ex: $\sum_{k=1}^{\infty} \frac{1}{2k^2 - k}$ COMPARE TO $\sum_{k=1}^{\infty} \frac{1}{k^2}$ P-SERIES $p=2$ $\therefore$ CONVERGES

$\rho = \lim_{k\to\infty} \frac{1}{2k^2-k} \cdot \frac{k^2}{1} = \frac{k^2}{2k^2-k} \longrightarrow 1/2$

$\therefore \sum_{k=1}^{\infty} \frac{1}{2k^2-k}$ CONVERGES

Ex: $\sum_{k=1}^{\infty} \frac{1}{k - \frac{1}{4}}$ COMPARE TO $\sum_{k=1}^{\infty} \frac{1}{k}$ HARMONIC $\therefore$ DIVERGES

$\rho = \lim_{k\to\infty} \frac{1}{k-1/4} \cdot \frac{k}{1} = \frac{k}{k-1/4} \to 1$

$\therefore \sum_{k=1}^{\infty} \frac{1}{k-1/4}$ DIVERGES

Ex: $\sum_{k=1}^{\infty} \frac{3k^3 - 2k^2 + 4}{k^5 - k^3 + 2}$ COMPARE TO $\sum_{k=1}^{\infty} \frac{1}{k^2}$ P-SERIES $p=2$ CONVERGES

$\rho = \lim_{k\to\infty} \frac{3k^3-2k^2+4}{k^5-k^3+2} \cdot \frac{k^2}{1} \longrightarrow 3$

$\therefore \sum \frac{3k^3-2k^2+4}{k^5-k^3+2}$ CONVERGES

Comparison Test

Let Σa_k and Σb_k have nonnegative terms such that:

$$a_1 \le b_1; a_2 \le b_2; a_3 \le b_3 \dots$$

1. If Σb_k converges, then Σa_k converges.
2. If Σa_k diverges, then Σb_k diverges.

If the "bigger" one converges, the "smaller" one converges.

If the "smaller" one diverges, the "bigger" one diverges.

Ex: $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k - \frac{1}{4}}$ COMPARE TO $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k}$ HARMONIC $\Rightarrow$ DIVERGES

PROVE THAT $\frac{1}{k}$ IS SMALLER THAN $\frac{1}{k-1/4}$

$k > k - 1/4$ FOR $k \ge 1$

$\frac{1}{k} < \frac{1}{k-1/4}$

$\therefore \sum \frac{1}{k-1/4}$ DIVERGES

Ex: $\displaystyle\sum_{k=1}^{\infty} \frac{1}{\sqrt[3]{k} - 5}$ COMPARE TO $\displaystyle\sum_{k=1}^{\infty} \frac{1}{\sqrt[3]{k}}$ P-SERIES $p = 1/3$ $\Rightarrow$ DIVERGES

$\sqrt[3]{k} > \sqrt[3]{k} - 5$

$\frac{1}{\sqrt[3]{k}} < \frac{1}{\sqrt[3]{k} - 5}$

$\therefore \displaystyle\sum_{k=1}^{\infty} \frac{1}{\sqrt[3]{k} - 5}$ DIVERGES

Ex: $\sum \frac{1}{\sqrt[3]{k} + 5}$

COMPARE TO $\sum \frac{1}{7\sqrt[3]{k}}$

$7\sqrt[3]{k} > \sqrt[3]{k} + 5$ IF $k \ge 1$

$\frac{1}{7\sqrt[3]{k}} < \frac{1}{\sqrt[3]{k} + 5}$

Ex: $\displaystyle\sum_{k=1}^{\infty} \frac{1}{2k^2 + k}$ COMPARE TO $\displaystyle\sum_{k=1}^{\infty} \frac{1}{2k^2}$ P-SERIES $p = 2$ CONVERGES

$2k^2 < 2k^2 + k$ FOR $k \ge 1$

$\frac{1}{2k^2} > \frac{1}{2k^2 + k}$

$\therefore \sum \frac{1}{2k^2 + k}$ CONVERGES

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6. Root Test
7. Limit Comparison Test
8. Comparison Test

Homework:

Anton 11.6 # 1 – 35 odd