



# AP Calculus BC

## CHAPTERS 8 & 10 WORKSHEET

TECHNIQUES OF INTEGRATION & DIFFERENTIAL EQUATIONS

Name \_\_\_\_\_

Seat # \_\_\_\_\_ Date \_\_\_\_\_

### Logistic Model

A GRAPHING CALCULATOR MAY BE USED FOR ALL QUESTIONS

1. Biologists stocked a lake with 400 trout and estimated the carrying capacity (the maximal population of trout in that lake) to be 10,000. Observation of the trout in the lake revealed that the number of trout tripled in the first year.
  - a) Assuming that the size of the trout population satisfies a logistic model, write the differential equation for the model and an expression for the size of the trout population,  $P$ , after  $t$  years. Use your expression and the information given to find the value of the constant  $k$ .
  - b) What will be the trout population when the rate of growth predicted by the logistic model changes from increasing to decreasing? Explain your answer.
  - c) Find the value of the largest rate of growth for the trout population. Show all your work.
  - d) After how many years will the rate of growth of the trout population change from increasing to decreasing? Show all your work.
  - e) Find  $\lim_{t \rightarrow \infty} P(t)$ .
  - f) Set up an equation to find when the trout population will reach 8,000. Use your calculator to solve your equation.
  - g) Suppose now that instead of stocking the lake with 400 trout, biologists decided to start up with 12,000 trout. Find  $\lim_{t \rightarrow \infty} P(t)$ . Write a sentence interpreting your answer.
  - h) If the initial trout population is 12,000 and the value of the constant  $k$  is the same as the one found in 1 a), find an expression for the size of the trout population,  $P$ , after  $t$  years.

2. The spread of flue in a certain school is given by the formula  $P(t) = \frac{2,100}{1 + e^{4.3-t}}$ , where  $t$  is the number of days after students are first exposed to infected students.
  - a) The formula is a solution of a logistic differential equation. Identify the values of both the constant  $k$  and the carrying capacity.
  - b) Find  $P(0)$ . Explain its meaning in the context of the problem.
  - c) How many days will it take for a total of 400 students to become infected? Show all your work.

3. Which ones of the following differential equations model logistic growth? If the equation models logistic growth, identify the values of both the constant  $k$  and the carrying capacity of the model.

a)  $\frac{dy}{dt} = \frac{t}{20} \cdot \left(1 - \frac{t}{5,000}\right)$

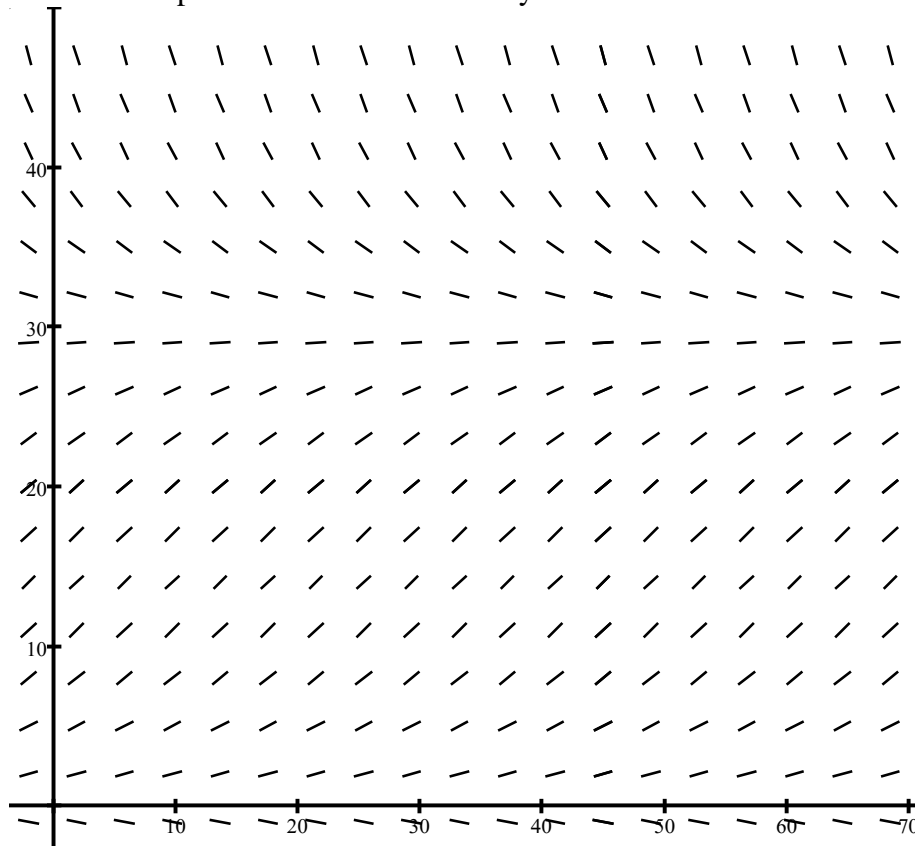
b)  $\frac{100}{P} \cdot \frac{dP}{dt} = \left(1 - \frac{P}{350}\right)$

c)  $\frac{dy}{dt} = 0.003 \cdot (y - 0.001y^2)$

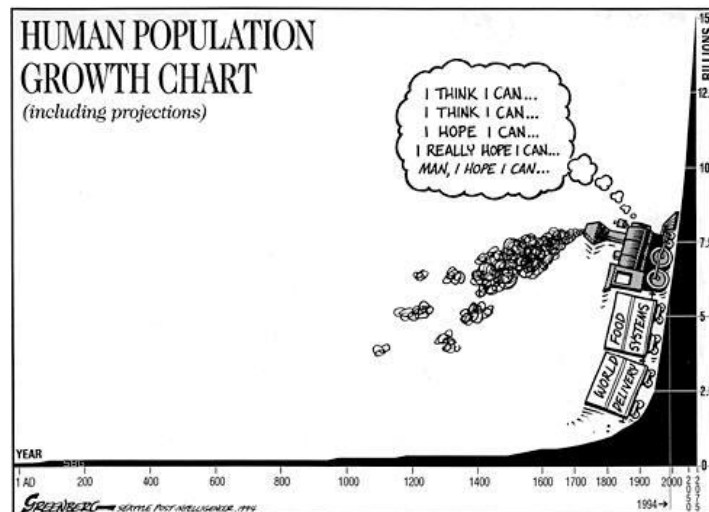
d)  $\frac{dP}{dt} = \frac{P}{45} \left(3 - \frac{P}{1,000}\right)$

SEE OTHER SIDE

4. The graph below represents a slope field for a logistic differential equation modeling the number of wolves in a national park. Time is measured in years.



- Use the graph to estimate the carrying capacity of the wolf population.
- Sketch the graph of the particular solution to the differential equation when 10 wolves are initially introduced in the park.
- Sketch the graph of the particular solution to the differential equation when 40 wolves are initially introduced in the park.
- Using your graph from question 4b), estimate the wolf population after 20 years. Use this number and your answer to question 4a) to find an expression for the wolf population,  $P$ , as a function of the number of years,  $t$ .





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# ANSWER KEY

1.

$$a) \frac{dP}{dt} = kP \cdot \left(1 - \frac{P}{10,000}\right)$$

$$P = \frac{10,000}{1 + Ae^{-kt}}$$

$$t = 0 \Rightarrow P = 400 \Rightarrow A = 24$$

$$t = 1 \Rightarrow P = 1,200 \Rightarrow k = \ln\left(\frac{36}{11}\right) \approx 1.186 \left. \vphantom{\begin{matrix} t = 0 \\ t = 1 \end{matrix}} \right\} \Rightarrow P = \frac{10,000}{1 + 24e^{-1.186t}}$$

b)  $P = 5,000$ . The point of inflection in the graph of a logistic model occurs when the population reaches half the value of the carrying capacity.

$$c) P = 5,000 \Rightarrow \frac{dP}{dt} = 1.186 \cdot (5,000) \cdot \left(1 - \frac{(5,000)}{10,000}\right) \approx 2964.059 \frac{\text{trout}}{\text{year}}$$

$$d) P = 5,000 \Rightarrow 5,000 = \frac{10,000}{1 + 24e^{-1.186t}} \Rightarrow t \approx 2.680 \text{ years}$$

$$e) \lim_{t \rightarrow \infty} P(t) = 10,000$$

$$f) P = 8,000 \Rightarrow 8,000 = \frac{10,000}{1 + 24e^{-1.186t}} \Rightarrow t \approx 3.850 \text{ years}$$

g)  $\lim_{t \rightarrow \infty} P(t) = 10,000$ . The carrying capacity is still equal to  $\lim_{t \rightarrow \infty} P(t)$ . However, when the initial population is smaller than the carrying capacity, the population continually increases. When the initial population is larger than the carrying capacity, the population continually decreases.

$$h) P = \frac{10,000}{1 + Ae^{-kt}}$$

$$t = 0 \Rightarrow P = 12,000 \Rightarrow A = -\frac{1}{6}$$

$$k = \ln\left(\frac{36}{11}\right) \approx 1.186 \left. \vphantom{\begin{matrix} t = 0 \\ k = \ln\left(\frac{36}{11}\right) \end{matrix}} \right\} \Rightarrow P = \frac{10,000}{1 - \frac{1}{6}e^{-1.186t}} = \frac{60,000}{6 - e^{-1.186t}}$$

2.

a)  $k = 1$ ; the carrying capacity is 2,100 (the value of  $A$  is  $e^{4.3} \approx 73.700$ .)

b)  $P(0) \approx 28.112$ . In the first day of the spread of the disease, about 28 students came to school with the flue. These students spread the disease to their school mates!

$$c) P = 400 \Rightarrow 400 = \frac{2,100}{1 + e^{4.3-t}} \Rightarrow t \approx 2.853 \text{ days}$$

3.

a)  $\frac{dy}{dt} = \frac{t}{20} \cdot \left(1 - \frac{t}{5,000}\right) \Rightarrow$   
 Not a logistic model

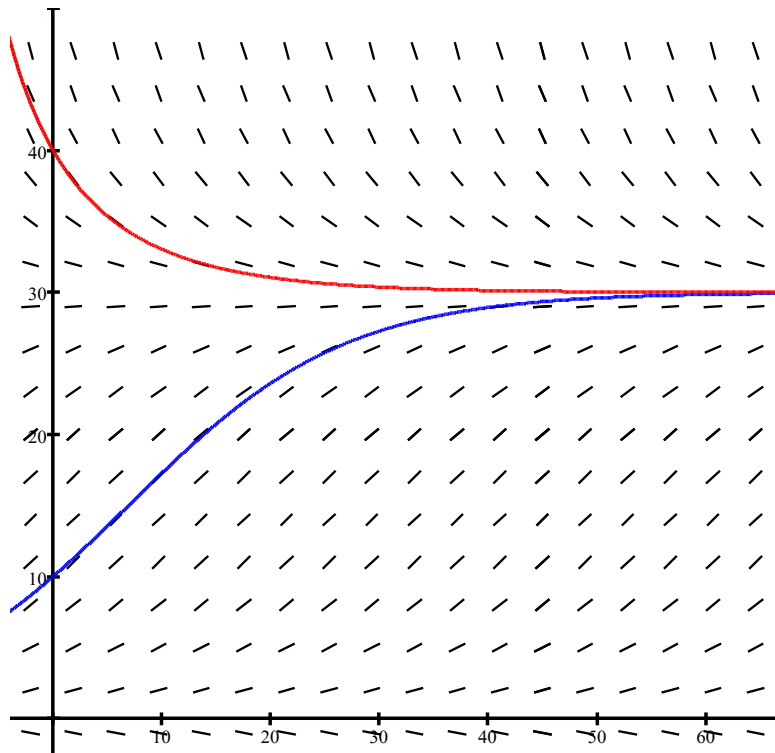
b)  $\frac{100}{P} \cdot \frac{dP}{dt} = \left(1 - \frac{P}{350}\right) \Rightarrow$   
 $\frac{dP}{dt} = \frac{P}{100} \cdot \left(1 - \frac{P}{350}\right) \Rightarrow$  Logistic model:  
 $k = \frac{1}{100}$  and  $M = 350$

c)  $\frac{dy}{dt} = 0.003 \cdot (y - 0.001y^2) \Rightarrow$   
 $\frac{dy}{dt} = 0.003y \cdot \left(1 - \frac{y}{1000}\right) \Rightarrow$   
 Logistic model:  
 $k = 0.003$  and  $M = 1,000$

d)  $\frac{dP}{dt} = \frac{P}{45} \left(3 - \frac{P}{1,000}\right) \Rightarrow$   
 $\frac{dP}{dt} = \frac{3P}{45} \left(1 - \frac{P}{3,000}\right) = \frac{P}{15} \left(1 - \frac{P}{3,000}\right) \Rightarrow$   
 Logistic model:  
 $k = \frac{1}{15}$  and  $M = 3,000$

4.

a) 30 wolves



b) and c) see above

d) From the graph, we can estimate that after 20 years there will be about 22 wolves.

$$P = \frac{30}{1 + Ae^{-kt}}$$

$$t = 0 \Rightarrow P = 10 \Rightarrow A = 2$$

$$t = 20 \Rightarrow P = 22 \Rightarrow k = \frac{1}{20} \ln\left(\frac{11}{2}\right) \approx 0.085 \left. \vphantom{t = 20} \right\} \Rightarrow P = \frac{30}{1 + 2e^{-0.085t}}$$