$$\int x(x^2+1)^{20}\,dx$$

$u = x^2+1$

$du = 2x\,dx$

$\frac{du}{2x} = dx$

$\int \cancel{x}\, u^{20} \frac{du}{2\cancel{x}}$

$\frac{1}{2}\int u^{20}\,du$

$\frac{1}{2}\cdot\frac{1}{21}u^{21}+C \Rightarrow \boxed{\frac{1}{42}(x^2+1)^{21}+C}$

LORENZ WAY

$\int x(x^2+1)^{20}\,dx$

$\frac{1}{2}\underbrace{\int 2x\,(x^2+1)^{20}\,dx}$

$\frac{1}{2}\,\frac{1}{21}(x^2+1)^{21}+C$

$$\int \sin 5x\,dx$$

$u = 5x$

$du = 5\,dx$

$\frac{du}{5} = dx$

$\frac{1}{5}\int \sin u\,du$

$-\frac{1}{5}\cos u + C$

$\boxed{-\frac{1}{5}\cos(5x)+C}$

$\int \sin kx\,dx = -\frac{1}{k}\cos kx + C$

$$\int e^{-3x}\,dx$$

$u = -3x$

$du = -3\,dx$

$-\frac{1}{3}\int e^u\,du$

$-\frac{1}{3}e^u + C$

$\boxed{-\frac{1}{3}e^{-3x}+C}$

$\int e^{kx}\,dx = \frac{1}{k}e^{kx}+C$

$$\int \sin^{2}4x\,\cancel{\cos(4x)}\,dx$$

$u = \sin 4x$

$du = 4\,\cancel{\cos 4x}\,dx$

$\frac{1}{4}\int u^2\,du$

$\frac{1}{4}\cdot\frac{1}{3}u^3 + C = \frac{1}{12}(\sin 4x)^3 + C$

$= \boxed{\frac{1}{12}\sin^3 4x + C}$

$$\int x\sqrt{x+1}\,dx$$

$u = x+1 \Rightarrow x = u-1$

$du = dx$

DOUBLE SUBSTITUTION

$\int x\sqrt{u}\,du$

$\int (u-1)u^{1/2}\,du$

$\int u^{3/2} - u^{1/2}\,du$

$\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2} + C = \boxed{\frac{2}{5}(x+1)^{5/2} - \frac{2}{3}(x+1)^{3/2} + C}$

$$\int_{-1}^{0} \frac{\cancel{x}}{x^2+5}\,dx$$

$u = x^2+5$

$du = 2\cancel{x}\,dx$

CHANGE LIMITS

$x=-1 \Rightarrow u = (-1)^2+5 = 6$

$x = 0 \Rightarrow u = (0)^2+5 = 5$

$$\frac{1}{2}\int_{6}^{5} \frac{1}{u}\,du$$

$$\frac{1}{2}\ln|u|\Big]_{6}^{5}$$

$$\frac{1}{2}(\ln 5 - \ln 6)$$

$$\boxed{\frac{1}{2}\ln(5/6)} = -\frac{1}{2}\ln(6/5) = \ln\sqrt{5/6} = -\ln\sqrt{6/5}$$

$$\int_{0}^{\ln 2} e^{-3x}\,dx = -\frac{1}{3}e^{-3x}\Big]_{0}^{\ln 2}$$

$\star\ \boxed{e^{\ln f(x)} = f(x)}$

$$= -\frac{1}{3}\left(e^{-3\ln 2} - e^{0}\right)$$

$$= -\frac{1}{3}\left(\cancel{e^{\ln}}\, 2^{-3} - 1\right)$$

$$= -\frac{1}{3}\left(\frac{1}{8} - 1\right)$$

$$= -\frac{1}{3}\left(-\frac{7}{8}\right)$$

$$= \boxed{\frac{7}{24}}$$