AP CALCULUS

Stuff you MUST know Cold

* topic only on BC

Basic Derivatives

where u is a function of x, and a is a constant.

$$\frac{d}{dx}\left(x^n\right)=nx^{n-1}$$

$$\frac{d}{dx}(\sin x)=\cos x$$

$$\frac{d}{dx}(\cos x)=-\sin x$$

$$\frac{d}{dx}(\tan x)=\sec^2 x$$

$$\frac{d}{dx}(\cot x)=-\csc^2 x$$

$$\frac{d}{dx}(\sec x)=\sec x\tan x$$

$$\frac{d}{dx}(\csc x)=-\csc x\cot x$$

$$\frac{d}{dx}(\ln u)=\frac{1}{u}\frac{du}{dx}$$

$$\frac{d}{dx}(\log_a x)=\frac{1}{x\ln a}$$

$$\frac{d}{dx}\left(e^u\right)=e^u\frac{du}{dx}$$

$$\frac{d}{dx}\left(a^{u(x)}\right)=a^{u(x)}\ln a\cdot\frac{du}{dx}$$

$$\frac{d}{dx}\left(\sin^{-1}u\right)=\frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$$

$$\frac{d}{dx}\left(\cos^{-1}x\right)=\frac{-1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\left(\tan^{-1}u\right)=\frac{1}{1+u^2}\cdot\frac{du}{dx}$$

$$\frac{d}{dx}\left(\cot^{-1}x\right)=\frac{-1}{1+x^2}$$

$$\frac{d}{dx}\left(\sec^{-1}u\right)=\frac{1}{|u|\sqrt{u^2-1}}\cdot\frac{du}{dx}$$

$$\frac{d}{dx}\left(\csc^{-1}x\right)=\frac{-1}{|x|\sqrt{x^2-1}}$$

Differentiation Rules

Chain Rule $\frac{d}{dx}\left[f(g(x))\right]=f'(g(x))g'(x)$

Product Rule $\frac{d}{dx}(uv)=u\frac{dv}{dx}+v\frac{du}{dx}$

Quotient Rule $\frac{d}{dx}\left(\frac{u}{v}\right)=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$

Some handy INTEGRALS:

$$\int\tan x\,dx=\ln|\sec x|+C$$

$$=-\ln|\cos x|+C$$

$$\int\sec x\,dx=\ln|\sec x+\tan x|+C$$

Curve sketching and analysis

$y=f(x)$ must be continuous at each:

critical point: $\frac{dy}{dx}=0$ or <u>undefined</u>

inc/dec. function $f(x)$: $f'>0,\ f'<0$

concavity ≡ inc/dec slope: $f''>0,\ f''<0$

local minimum:

$\frac{dy}{dx}$ goes (–,0,+) or (–,DNE,+) or $\frac{d^2y}{dx^2}>0$

local maximum:

$\frac{dy}{dx}$ goes (+,0,–) or (+,DNE,–) or $\frac{d^2y}{dx^2}<0$

point of inflection:

$f''=0$ or DNE <u>AND</u> concavity changes

$\frac{d^2y}{dx^2}$ goes from (+ to –), (– to +),

Abs. Max/Min: eval. crit # & endpts.

OR discuss "always inc or always dec."

Intermediate Value Theorem:

If the function $f(x)$ is <u>continuous</u> on $[a, b]$, for all k between $f(a)$ and $f(b)$, there exists at least one number $x=c$ in the open interval (a, b) such that $f(c)=k$.

Extreme Value Theorem:

If the function $f(x)$ is <u>continuous</u> on $[a, b]$, then there exists an <u>absolute</u> max and min on that interval.

Rolle's Theorem:

If the function $f(x)$ is <u>continuous</u> on $[a, b]$, AND differential on the interval (a, b), AND $f(a)=f(b)$, then there is at least one number $x=c$ in (a, b) such that $f'(c)=0$

Mean Value Theorem: $m_{\text{secant}}=m_{\text{tangent}}$

If the function $f(x)$ is <u>continuous</u> on $[a, b]$, AND <u>differential</u> on the interval (a, b), then there is at least one number $x=c$ in (a, b) such that $f'(c)=\frac{f(b)-f(a)}{b-a}$.

MVT of Integrals i.e. AVERAGE VALUE:

If the function $f(x)$ is <u>continuous</u> on $[a, b]$ and <u>differential</u> on the interval (a, b), then there exists at least one number $x=c$ on (a, b) such that

$$f(c)(b-a)=\int_a^b f(x)dx$$

$$Area_{\text{rectangle}}=Net\ Area_{\text{Integral}}$$

$$f(c)=\frac{1}{(b-a)}\int_a^b f(x)dx.$$

This value f(c) is the "average value" of the function on the interval [a, b].

Limit Strategies: Factor and cancel, Rationalize Numerator, u-sub, HA rules:

2 HA

$$\lim_{x\to\infty}\frac{a\,x}{\sqrt{bx^2+c}}=\frac{a}{\sqrt{b}}$$

$$\lim_{x\to-\infty}\frac{a\,x}{\sqrt{bx^2+c}}=\frac{-a}{\sqrt{b}}$$

To find all HA: Take limit as $x\to both\pm\infty$

Approximation Methods for Integration

Use Geometry formulas

Rectangles - Left, Right and Middle Riemann Sums A = bh

Trapezoids: $A=\frac{1}{2}(b_1+b_2)h$

(Effects of inc/dec & cocavity on approx.)

Concave up: M under estimate, T over estimate

Concave down: M over estimate, T under estimate

Inc: L=under, R=over. Dec: L = over, R = under

First Fundamental Th. of Calculus

$$\int_a^b f'(x)dx=f(b)-f(a)$$

2nd Fundamental Th. of Calculus

$$\frac{d}{dx}\int_{a(x)}^{b(x)}f(t)dt=$$

$$f(b(x))\,b'(x)-f(a(x))\,a'(x)$$

Solids of Revolution and friends

Disk Method

$$V=\pi\int_a^b\left[R(x)\right]^2dx$$

Washer Method

$$V=\pi\int_a^b\left(\left[R(x)\right]^2-\left[r(x)\right]^2\right)dx$$

volume by cross section (not rotated)

$$V=\int_a^b Area(x)\,dx$$

$$A_{eq.\,lat.\,\Delta}=\frac{s^2}{4}\sqrt{3}$$

*Arc Length $s=\int_a^b\sqrt{1+\left[f'(x)\right]^2}\,dx$

*Surface Area

$$SA=2\pi\int_a^b radius\sqrt{1+\left[f'(x)\right]^2}\,dx$$

Distance, Velocity, and Acceleration

velocity = $\frac{d}{dt}$(position)

acceleration = $\frac{d}{dt}$(velocity)

*velocity vector = $\left\langle\frac{dx}{dt},\frac{dy}{dt}\right\rangle$

speed = $\|v\|=\sqrt{(x')^2+(y')^2}$ *

displacement = $\int_{t_o}^{t_f}v\,dt$

total distance = $\int_{\text{initial time}}^{\text{final time}}|v|\,dt$

$$\int_{t_o}^{t_f}\sqrt{(x')^2+(y')^2}\,dt \ *$$

Av. velocity = $\frac{1}{b-a}\int_a^b v(t)dt=\frac{s(b)-s(a)}{b-a}$

$$\frac{1}{b-a}\int_a^b\sqrt{\left[x'(t)\right]^2+\left[y'(t)\right]^2}\,dt \ *$$

BC TOPICS and important TRIG identities and values:

l'Hôpital's Rule

If $\dfrac{f(a)}{g(b)} = \dfrac{0}{0}$ or $= \dfrac{\infty}{\infty}$,

then $\lim\limits_{x\to a}\dfrac{f(x)}{g(x)} = \lim\limits_{x\to a}\dfrac{f'(x)}{g'(x)}$

Euler's Method

If given that $\frac{dy}{dx} = f(x, y)$ and the solution passes through (x_o, y_o),

$$x_{\text{new}} = x_{\text{old}} + \Delta x$$

Basically Algebra I:
$y = m(x - x_1) + y_1$

$$y_{\text{new}} = \left(\left.\frac{dy}{dx}\right|_{(x_{\text{old}}, y_{\text{old}})} \right)(\Delta x) + y_{\text{old}}$$

Integration by Parts

(u=ILATE=dv)

$$\int u\,dv = uv - \int v\,du$$

$$\int f(x)g'(x)dx = f(x)g(x) - \int g(x)f'(x)dx$$

Integral of Ln

Use integration by parts and let $u = \ln x$

$$\int \ln x\,dx = x\ln x - x + C$$

Taylor Series

If the function f is "smooth" *(continuous and differentiable)* at $x = c$, then it can be approximated by the n^{th} degree polynomial

$$f(x) \approx f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \ldots + \frac{f^{(n)}(c)}{n!}(x-c)^n.$$

Maclaurin Series *(Taylor Series about x = 0)*

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} + \ldots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots + \frac{x^{2n}(-1)^n}{(2n)!} + \ldots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + \frac{x^{2n+1}(-1)^n}{(2n+1)!} + \ldots$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots + x^n + \ldots$$

$$\ln(x) = \frac{(x-1)^1}{1} - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \ldots + \frac{(x-1)^n(-1)^{n+1}}{n} + \ldots$$

Slope of a Parametric equation

Given $x(t)$ and $y(t)$, the slope is

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}, \quad \frac{d^2y}{dx^2} = \frac{D_t\left(\frac{dy}{dt}\right)}{\frac{dx}{dt}}$$

Polar Curve

For a polar curve $r(\theta)$, the

AREA inside a "leaf" = $\frac{1}{2}\int_{\theta_1}^{\theta_2}\left[r(\theta)\right]^2 d\theta$

θ_1 and θ_2 are the "first" two times that $r = 0$.

$\dfrac{dr}{d\theta}$ determines inc/dec and relative max/mins.

The **SLOPE** of $r(\theta)$ at a given θ is

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{d}{d\theta}\left[r(\theta)\sin\theta\right]}{\frac{d}{d\theta}\left[r(\theta)\cos\theta\right]}$$

nth Term: diverges if $\lim\limits_{n\to\infty} a_n \neq 0$

Ratio Test: converges if $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| < 1$

If the limit equals 1, you know nothing so check the endpoints using another test.

Root Test: converges if $\lim\limits_{n\to\infty}\sqrt[n]{a_n} < 1$

***p*–series Test:** $\sum\limits_{n=1}^{\infty}\dfrac{1}{n^p}$ converges if p > 1

Alternating: converges if alt. & $\lim\limits_{n\to\infty} a_n = 0$

Geometric Series: $\sum\limits_{n=0}^{\infty} a_1 r^n$ converges if $|r|<1$

Limit Comparison: If $\lim\limits_{n\to\infty}\dfrac{a_n}{b_n}$ exists and $\neq 0$, what one series does (*C or D*), so does the other.

Direct Comparison: If $0 < a_n < b_n$ and

if $\sum\limits_{n=1}^{\infty} a_n$ diverges, then $\sum\limits_{n=1}^{\infty} b_n$ diverges;

if $\sum\limits_{n=1}^{\infty} b_n$ converges then $\sum\limits_{n=1}^{\infty} a_n$ converges.

Error Bound (*Remainder*)

Alternating Series:

If $S_N = \sum\limits_{k=1}^{N}(-1)^n a_n$ is the N^{th} partial sum of a convergent alternating series,

$|R_N| \leq |a_{N+1}|$ *(the next term)*

Lagrange Error Bound of a Taylor Series:

Let c = # centered on and x = value you want to approximate. There exists a z between x and c,

such that $|R_n| \leq \dfrac{f^{n+1}(z)}{(n+1)!}|x-c|^{n+1}$

Find interval of $f^{n+1}(z)$ to find error interval.

Values of Trigonometric Functions for Common Angles

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$
0°	0	1	0
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
$\frac{\pi}{2}$	1	0	"∞"
π	0	–1	0

*Know both the **inverse trig** and the **trig** values.*

Careful: $\tan\left(\frac{3\pi}{4}\right) = -1$ *but* $\arctan(-1) = -\frac{\pi}{4}$

Trig Identities

Double Angle

$\sin 2x = 2\sin x\cos x$

$\cos 2x = \cos^2 x - \sin^2 x = 1 - 2\sin^2 x$

Power Reduction

$\sin^2 x = \dfrac{(1-\cos 2x)}{2}$

$\cos^2 x = \dfrac{(1+\cos 2x)}{2}$

Pythagorean

$\sin^2 x + \cos^2 x = 1$

(others are easily derivable by dividing by $\sin^2 x$ or $\cos^2 x$)

$1 + \tan^2 x = \sec^2 x$

$\cot^2 x + 1 = \csc^2 x$

Reciprocal

$\sec x = \dfrac{1}{\cos x}$ *or* $\cos x\sec x = 1$

$\csc x = \dfrac{1}{\sin x}$ *or* $\sin x\csc x = 1$

Odd-Even

$\sin(-x) = -\sin x$ (odd)
$\cos(-x) = \cos x$ (even)
$\tan(-x) = -\tan x$ (odd)

Infinite Sums

Geometric Series: $S_\infty = \dfrac{a_1}{1-r}$ if $|r|<1$

Telescoping Series: Expand & cancel

Special Series: $\sum\limits_{n=0}^{\infty}\dfrac{1}{n!} = e^1$