Chapter 4

Section 4.2 – Some Existence Theorems
Find the coordinates of all relative extrema:

\[ x^2 - xy + y^2 = 3 \]

\[ 2x - y \frac{dy}{dx} + y = 0 \]

\[ \Rightarrow \frac{dy}{dx} = \frac{y-2x}{2y-x} \]

\[ 1. \quad \frac{dy}{dx} = 0 \]

\[ y-2x = 0 \]

\[ 2y-x = 0 \]

\[ y = 2x \]

\[ 2y = x \]

\[ 2. \quad \frac{dy}{dx} = \phi \]

\[ (2y)^2 - (2y)y + y^2 = 3 \]

\[ 3y^2 = 3 \]

\[ y = \pm 1 \]

\[ (1, 2), (-1, -2), (2, 1), (-2, -1) \]
Find the coordinates of all relative extrema (cont.):

\[ x^2 - xy + y^2 = 3 \]

\[ \frac{dy}{dx} = \frac{y-2x}{2y-x} \]

\[ \frac{d^2y}{dx^2} = \frac{(2y-x)\left(\frac{dy}{dx}\right) - (y-2x)(2 \frac{dy}{dx} - 1)}{(2y-x)^2} \]

\[ \left. \frac{d^2y}{dx^2} \right|_{(1,2)} = \frac{(2-1)(0-2) - (2-2)(2-1)}{(2-2-1)^2} = \frac{-6}{9} < 0 \]

\( \therefore \text{rel. max } @ (1,2) \)

\[ \left. \frac{d^2y}{dx^2} \right|_{(-1,-2)} = \frac{(-4+1)(0-2) - (-2+2)(2-1)}{(-4+1)^2} = \frac{6}{9} > 0 \]

\( \therefore \text{rel. min } @ (-1,-2) \)

\[ \left. \frac{dy}{dx} \right|_{(2,1)} \text{ DNE since } \left. \frac{dy}{dx} \right|_{(2,1)} \text{ DNE.} \]
Graphical Support:
Absolute (Global) Extrema

\[ \text{"FOR ALL"} \]

1. If \( f(c) \geq f(x) \quad \forall x \in \text{domain of } f \Rightarrow f(c) \text{ is max value of } f(x) \).
   
   \text{Absolute max occurs at } x = c.

2. If \( f(c) \leq f(x) \quad \forall x \in \text{domain of } f \Rightarrow f(c) \text{ is min value of } f(x) \).
   
   \text{Absolute min occurs at } x = c.
Intermediate Value Theorem (IVT)

1. If $f$ is cont on $[a,b]$ then $f(x)$ must take on all values between $f(a)$ and $f(b)$.

2. If $f$ is cont. on $[a,b]$ $\Rightarrow \forall L \in [f(a), f(b)]$ \exists $c \in (a,b)$ such that $f(c) = L$. 

\[ f(b) \quad \overset{L}{\longrightarrow} \quad f(a) \]

\[ a \quad c \quad b \]
Extreme Value Theorem (EVT)

If \( f(x) \) is cont on \([a, b]\) \( \Rightarrow \) \( f(x) \) has a maximum value and a minimum value for \( x \in [a, b] \)

\[\begin{align*}
\text{Max value} &= f(b) \\
\text{Min value} &= f(a)
\end{align*}\]

Max/Min values occur at:

1. Endpoint, or
Let $g$ be a continuous function on the closed interval $[-2, 4]$. A few values of $g$ are given in this table:

<table>
<thead>
<tr>
<th>$x$</th>
<th>-2</th>
<th>0</th>
<th>2</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$g(x)$</td>
<td>1</td>
<td>-2</td>
<td>3</td>
<td>-5</td>
</tr>
</tbody>
</table>

Which intervals must contain a solution to $g(x) = -1$?

1. $[-2, 0]$
2. $[0, 2]$
3. $[2, 4]$
Find the maximum and minimum values of the function on the given interval.

\( f(x) = -x^3 + 12x + 5, \quad [-3, 3] \)

poly. fxn cont. m \([-3, 3]\)

\[ f'(x) = -3x^2 + 12 = 0 \]

\[ 3x^2 = 12 \]

\[ x^2 = 4 \]

\[ x = \pm 2 \]

\[ f(-3) = -4 \quad \text{max value is } -4 \]
\[ f(-2) = -11 \quad \text{max value is } -11 \]
\[ f(2) = 21 \quad \text{min value is } 21 \]
\[ f(3) = 14 \quad \text{min value is } 14 \]
Mean Value Theorem (MVT)

If $f$ is continuous on $[a, b]$ and differentiable on $(a, b)$, then:

$$
\exists \ c \in (a, b) \text{ s.t. } m_{\text{TAN}} \Big|_{x=c} = m_{\text{SEC}} [a, b]
$$

$$
f'(c) = \frac{f(b)-f(a)}{b-a}
$$
The graph of function $f$ has a vertical tangent at $x = 2$.

The graph of function $g$ has a sharp turn at $x = 1$.

$f$ not contin. @ $x = 2$

$g$ not diff. @ $x = 1$
Rafael said that since \( \frac{g(1) - g(0)}{1 - 0} = -5 \) there must be a number \( c \) in the interval \([0, 1]\) for which \( g'(c) = -5 \).

Which condition makes Rafael’s claim true?

1. \( g \) cont on \([0, 1]\)
2. \( g \) diff on \((0, 1)\)
Let \( g(x) = x^3 - 16x \) and let \( c \) be the number that satisfies the Mean Value Theorem for \( g \) on the interval \([-4, 2]\).

What is \( c \)?

\[
g'(c) = \frac{g(2) - g(-4)}{2 - (-4)}
\]

\[
g'(x) = 3x^2 - 16
\]

\[
3x^2 - 16 = \frac{9(2) - 9(-4)}{2 - (-4)}
\]

\[
3x^2 - 16 = \frac{54}{6} = 9
\]

\[
3x^2 = 25
\]

\[
3x^2 = 25
\]

\[
x^2 = \frac{25}{3}
\]

\[
x = \pm \frac{5}{\sqrt{3}}
\]

\[
X = 2 \text{ is an endpoint and } C \in (-4, 2)
\]
Find the value that satisfies the MVT on the given interval.

\[ f(x) = x^2 + x, \quad [-4, 6] \]

\[ f(x) = \frac{1}{x-1}, \quad [0, 3] \]

\[ f = (x-1)^{-1} \]

\[
\frac{f(3) - f(0)}{3 - 0} = \frac{\frac{1}{3} + 1}{3} = \frac{1}{2}
\]

\[
f'(x) = -(x-1)^{-2} = \frac{1}{2}
\]

\[
\frac{1}{(x-1)^2} = \frac{1}{2}
\]

\[
-(x-1)^2 = \frac{1}{2}
\]

\[
-(x-1)^2 = a
\]

MVT does not apply if not continuous at \( x = 1 \).
Homework/Classwork:

AP Packet #1-4, 14, 25-29 odd, 37-40