Power Series

Anton 11.8

Power Series in $x$: a series of the form

$$\sum_{k=0}^{\infty} c_k x^k = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$$

Examples:

$$\sum_{k=0}^{\infty} x^k = 1 + x + x^2 + x^3 + \cdots \quad \text{Geometric, } r = x$$

$$\sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$
We would like to find for what values of $x$ the series converges. What test should we use?

**Ratio Test for Absolute Convergence**

Let

$$\rho = \lim_{k \to \infty} \frac{|u_{k+1}|}{|u_k|}$$

1. If $\rho < 1$ then the series converges absolutely.
2. If $\rho > 1$ then the series diverges.
3. If $\rho = 1$ then the test is inconclusive*.

*Further examination is required.

Example: Find the interval and radius of convergence.

Use Ratio Test

$$\sum_{k=0}^{\infty} x^k \Rightarrow \rho = \lim_{k \to \infty} \left| \frac{x^{k+1}}{x^k} \right| = |x| < 1$$

$$\therefore |x| < 1 \Rightarrow \{-1 < x < 1\}$$

If $x = \pm 1$, $\rho = 1 \Rightarrow$ further examination.

Check endpoints:

$x = -1: \sum_{k=0}^{\infty} (-1)^k = 1 - 1 + 1 - 1 \ldots \text{diverges}$

$x = 1: \sum_{k=0}^{\infty} (1)^k = 1 + 1 + 1 \ldots \text{diverges}$

Interval of convergence: $-1 < x < 1$ or $(-1, 1)$, or $|x| < 1$

Radius of convergence: $1$
Example: Find the interval and radius of convergence.

\[ \sum_{k=0}^{\infty} \frac{x^k}{k!} \Rightarrow \rho = \lim_{k \to \infty} \left| \frac{x^{k+1}}{(k+1)!} \cdot \frac{x}{x^k} \right| = \left| \frac{x}{k+1} \right| = 0 < 1 \]

\[ \therefore \sum_{k=0}^{\infty} \frac{x^k}{k!} \text{ converges for all } x \Rightarrow \text{ INTERVAL } = (-\infty, \infty) \]

\text{RADIUS of conv } = \infty

Example: Find the interval and radius of convergence.

\[ \sum_{k=0}^{\infty} k!x^k \Rightarrow \rho = \lim_{k \to \infty} \left| \frac{(k+1)x^{k+1}}{(k+1)!} \cdot \frac{k}{x^k} \right| = \left| \frac{(k+1)x}{x} \right| \to \infty > 1 \]

\[ \therefore \sum_{k=0}^{\infty} k!x^k \text{ diverges for } x \neq 0 \]

\[ \text{or } \sum_{k=0}^{\infty} k!x^k \text{ converges for } x=0 \text{, (RADIUS } = 0) \]
Example: Find the interval and radius of convergence.

\[ \sum_{k=0}^{\infty} (-1)^k \frac{x^k}{3^k(k+1)} \Rightarrow \rho = \lim_{k \to \infty} \left| \frac{\frac{x^{k+1}}{3^{k+1}(k+2)}}{\frac{x^k}{3^k(k+1)}} \right| = \left| \frac{x}{3} \right| < 1 \]

\[ \left| \frac{x}{3} \right| < 1 \Rightarrow -1 < \frac{x}{3} < 1 \Rightarrow -3 < x < 3 \)

CHECK LIMIT:

\[ x = -3 : \sum \frac{(-1)^k 3^k}{3^k(k+1)} = \sum \frac{1}{k+1} \rightarrow \text{DIV. HARMONIC} \]

\[ x = 3 : \sum \frac{(-1)^k 3^k}{3^k(k+1)} = \sum \frac{(-1)^k}{k+1} \rightarrow \text{CONT. HARMONIC} \]

\[ \text{INTERVAL OF CONV} : -3 < x \leq 3 \text{ or } (-3, 3] \]

\[ \text{RADIUS OF CONV} : 3 \]

Convergence/Divergence of a Power Series in \( x \): (Centered at \( x = c \))

Exactly one of the following is true:

1. Series converges only for \( x = 0 \).
2. Series converges absolutely for all \( x \).
3. Series converges for \( x \) in an interval \((-R, R)\) and diverges everywhere else. At \( R \) or \(-R\), the series may converge absolutely, conditionally, or it may diverge – you need to check each endpoint.

\( R \) is called the radius of convergence.
Power Series in \((x - a)\): a series of the form

\[
\sum_{k=0}^{\infty} c_k (x - a)^k = c_0 + c_1 (x - a) + c_2 (x - a)^2 + \cdots
\]

Example:

\[
\sum_{k=0}^{\infty} \frac{(x - 5)^k}{k^2} = (x - 5) + \frac{(x - 5)^2}{2} + \frac{(x - 5)^3}{3^2} + \cdots
\]

Find the interval and radius of convergence.

\[
\sum_{k=0}^{\infty} \frac{(x - 5)^k}{k^2} \Rightarrow \rho = \lim_{k \to \infty} \frac{(x - 5)^k}{(k+1)^2} = |x - 5| < 1
\]

\[|x - 5| < 1 \Rightarrow -1 < x - 5 < 1 \Rightarrow 4 < x < 6\]

\[\text{Check ends:}
\]

\[x = 4 : \sum \frac{(-1)^k}{k^2} \rightarrow \text{conv. for } p > 2
\]

\[x = 6 : \sum \frac{1}{k^2} \rightarrow \text{conv. for } p \rightarrow \infty
\]

\[\text{INT. OF CONV.: } 4 \leq x \leq 6 \quad \text{or } [4, 6]
\]

\[\text{RADIUS} = 1
\]
Convergence/Divergence of a Power Series in \((x - a)\):

Exactly one of the following is true:

1. Series converges only for \(x = a\).
2. Series converges absolutely for all \(x\).
3. Series converges for \(x\) in an interval \((a-R, a+R)\) and diverges everywhere else. The series may converge absolutely, conditionally, or it may diverge at the endpoints of the interval – you need to check.

\(R\) is called the radius of convergence.

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Homework:

Anton 11.8 # 1 – 23 odd