Infinite Series

Anton 11.3

A sequence of partial sums, \( \{S_n\} \), is defined as:

\[
\{S_n\} = s_1, s_2, s_3, \ldots \quad \text{where}
\]

\[
s_1 = a_1 \\
 s_2 = a_1 + a_2 \\
 s_3 = a_1 + a_2 + a_3 \\
 \vdots
s_n = a_1 + a_2 + a_3 + \cdots + a_n
\]

For our problem:

\[
s_1 = \frac{3}{10} = .3
\]

\[
s_2 = \frac{3}{10} + \frac{3}{100} = .33
\]

\[
s_3 = \frac{3}{10} + \frac{3}{100} + \frac{3}{1000} = .333
\]

\[
\therefore \{S_n\} = 0.3, 0.33, 0.333, 0.3333, \ldots \rightarrow 0.333\ldots = \frac{1}{3}
\]

\[
\therefore \sum_{k=1}^{\infty} \frac{3}{10^k} = \frac{1}{3}
\]
More formally:

Let \( \{S_n\} \) be a sequence of partial sums of \( \sum a_k \).

If \( \{S_n\} \to L \) then \( \sum a_k = L \)

If \( \{S_n\} \) diverges then \( \sum a_k \) diverges.

Ex: \( \sum (-1)^{k+1} = 1 - 1 + 1 - 1 + 1 - 1 + \ldots \)

\( S_n \) is \( 1, 0, 1, 0, 1, 0, \ldots \) \( \Rightarrow \) diverges \( \Rightarrow \) diverges

Identify \( a \) (the first term) and \( r \) (the common ratio):

\[
\begin{array}{c|c|c}
\sum_{k=1}^n 2^{k-1} &= 1 + 2 + 2^2 + \ldots & a = 1 \\
& & r = 2 \\
\sum_{k=1}^n \frac{3}{10^{k-1}} &= 1 - 1 + 1 - 1 + \ldots & a = \frac{1}{10} \\
& & r = -\frac{1}{10} \\
\sum_{k=1}^n \left(\frac{1}{2}\right)^k &= \frac{\frac{1}{2}}{1 - \frac{1}{2}} & a = \frac{1}{2} \\
& & r = -\frac{1}{2}
\end{array}
\]

Geometric Series: a series of the form

\[
\sum_{k=1}^{\infty} ar^{k-1} = a + ar + ar^2 + ar^3 + \ldots
\]

When do you think a geometric series will converge?

**Convergence of a Geometric Series:**

A geometric series will **converge** if \( |r| < 1 \).

If the geometric series **converges**, then the sum is:

\[
\sum_{k=1}^{\infty} ar^{k-1} = \frac{a}{1 - r}
\]

A geometric series **diverges** if \( |r| \geq 1 \).
Some other examples:

\[ \sum_{k=1}^{\infty} \frac{5}{4^k-1} = 5 + \frac{5}{4} + \frac{5}{4^2} + \cdots \]

\[ a = \frac{5}{4}, \quad r = \frac{1}{4} \implies \text{Conv. since } \frac{1}{4} < 1 \]

\[ \sum_{k=1}^{\infty} \frac{3}{(-7)^{k-1}} = 3 \left( -\frac{1}{7} + \frac{1}{7^2} + \cdots \right) \]

\[ a = \frac{3}{7}, \quad r = -\frac{1}{7} \implies \text{Conv. since } \left| -\frac{1}{7} \right| < 1 \]

Some other examples:

\[ 0.7 = \frac{7}{10} + \frac{7}{100} + \frac{7}{1000} + \cdots = \frac{7}{10} \left( 1 + \frac{1}{10} + \frac{1}{10^2} + \cdots \right) = \frac{7}{10} + \frac{\frac{7}{10}}{1 - \frac{1}{10}} = \frac{7}{9} \]

\[ a = \frac{7}{10}, \quad r = \frac{1}{10} \implies \text{Conv. since } \left| \frac{1}{10} \right| < 1 \]

Telescoping Series – an example

\[ \sum_{k=1}^{\infty} \frac{1}{k(k+1)} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots \]

Does this series converge?

No!

Harmonic Series

\[ \sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots \]

Telescoping Series – the terms will only cancel out if they are getting smaller.
Homework:

Anton 11.3 # 3 – 29 odd