Section 10.1

Calculus of Parametric Functions
Derivative at a point – parametrically: a parametric curve $x = x(t), \ y = y(t)$ has a derivative at $t_0$ if $x(t)$ and $y(t)$ have a derivative at $t_0$.

The curve is **differentiable** if it is differentiable at all values of $t$.

The curve is **smooth** if $x'(t)$ and $y'(t)$ are continuous and not simultaneously zero.
The formula for finding the slope of a parametric curve is:

\[
\frac{dy}{dx} = \frac{d}{dt} \frac{y}{dx} \left( y \right) = \frac{d}{dx} \left( \frac{y}{dx} \right) = \frac{dy}{dx} \frac{dt}{dt} = \frac{dy}{dt} \frac{x'(t)}{y'(t)}
\]

This makes sense if we think about cancelling \( dt \).

We assume that the denominator is not zero.
Find the equation of the line tangent to $x = t^2$, $y = t^3$ at $(1,1)$.

$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{3t^2}{2t} = \frac{3}{2} t \quad \Rightarrow \quad \frac{dy}{dx}\bigg|_{t=1} = \frac{3}{2} = m_{\text{tan}}$$

$$\therefore \quad (y-1) = \frac{3}{2} (x-1) \quad \Rightarrow \quad y = \frac{3}{2} x - \frac{1}{2}$$

Check: Convert to Cartesian coordinates, find $dy/dx$

$$x^{1/2} = y^{1/3} \quad \Rightarrow \quad x^3 = y^2 \quad \Rightarrow \quad 3x^2 = 2y \frac{dy}{dx}$$
\[ 3x^2 = 2y \frac{dy}{dx} \]

\[ \frac{3x^2}{2y} = \frac{dy}{dx} \]

\[ \left. \frac{dy}{dx} \right|_{(1,1)} = \frac{3(1)^2}{2(1)} = \frac{3}{2} \]
To find the second derivative of a parametric curve, we find the derivative of the first derivative:

\[ \frac{d^2 y}{dx^2} = \frac{d}{dx} \left( \frac{dy}{dx} \right) = \frac{d}{dt} \left( \frac{dy}{dx} \right) \]

1. Find the first derivative \((dy/dx)\).
2. Find the derivative of \((dy/dx)\) with respect to \(t\). \textbf{NUMERATOR}
3. Divide by \(dx/dt\). \textbf{DENOMINATOR}
Find the second derivative for \( x = t^2, \ y = t^3 \) at (1,1)

\[
\frac{d^2 y}{dx^2} = \frac{d}{dt} \left( \frac{dy}{dx} \right) = \frac{3}{2t} = \frac{3}{4t}
\]

\[
\Rightarrow \frac{d^2 y}{dx^2} \bigg|_{t=1} = \frac{3}{4t} \bigg|_{t=1} = \frac{3}{4}
\]

\[
\therefore \text{GRAPH SHOULD BE CCW AT } (1,1).
\]

\( t = 1 \)
Use the parametric equations to the right to answer the following questions.

1. Find the equation of the tangent line when \( t = 2 \).

   **Point:** \((x(2), y(2)) = (-2, -6)\)

   **Slope:** 
   \[
   \frac{dy}{dx} \bigg|_{t=2} = \frac{y(t)}{x(t)} \bigg|_{t=2} = \frac{1-3t^2}{1-2t} \bigg|_{t=2} = \frac{-11}{-3}
   \]

2. Find the second derivative and determine the concavity of the curve when \( t = 2 \).

   \[
   \frac{d^2y}{dx^2} = \frac{(1-2t)(-6t)-(1-3t^2)(-2)}{(1-2t)^2} \frac{(1-2t)}{(1-2t)} = \frac{(1-4)(12)-(1-1)(-2)}{(1-4)^3} = \frac{14}{-27} < 0
   \]

   \[
   \Rightarrow \frac{d^2y}{dx^2} \bigg|_{t=2} = \frac{(1-4)(-12)-(1-1)(-2)}{(1-4)^3} = \frac{14}{-27} < 0
   \]

   **Curve is concave down at** \( t = 2 \).
Use the parametric equations to the right to answer the following questions.

3. Find all values of $t$ for which the graph has horizontal and vertical tangents.

**Horizontal:** $\frac{dy}{dx} = 0 \Rightarrow y'(t) = 0$  \hspace{0.5cm} \left( x'(t) \neq 0 \right)

$y' = 1 - 3t^2 = 0$

$3t^2 = 1$

$t^2 = \frac{1}{3}$

$t = \pm \sqrt{\frac{1}{3}}$

**Vertical Tangents:** $\frac{dx}{dy}$ is undefined $\Rightarrow x'(t) = 0$

$x' = 1 - 2t = 0$

$t = \frac{1}{2}$
Use the parametric equations to the right to answer the following questions.

3. Find all values of \( t \) for which the graph has horizontal and vertical tangents.

\[
\begin{align*}
\text{x} &= t - t^2 \\
\text{y} &= t - t^3
\end{align*}
\]
A prolate cycloid is given by the given parametric equations. The graph crosses itself at (0,2). Find the equations of both tangent lines at this point.

\[ x = 2t - \pi \sin t \]
\[ y = 2 - \pi \cos t \]

\[
\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{\pi \sin t}{2 - \pi \cos t}
\]

\[
\left. \frac{dy}{dx} \right|_{t=\frac{\pi}{2}} = \frac{\pi \sin \frac{\pi}{2}}{2 - \pi \cos \frac{\pi}{2}} = \frac{\pi}{2}
\]

\[
\left. \frac{dy}{dx} \right|_{t=-\frac{\pi}{2}} = \frac{\pi \sin -\frac{\pi}{2}}{2 - \pi \cos -\frac{\pi}{2}} = -\frac{\pi}{2}
\]

\[
\begin{align*}
y &= \pm \frac{\pi}{2} x + 2
\end{align*}
\]
A prolate cycloid is given by the given parametric equations. The graph crosses itself at (0,2). Find the equations of both tangent lines at this point.

\[ x = 2t - \pi \sin t \]
\[ y = 2 - \pi \cos t \]
Practice:

p. 518 # 4, 6

Homework:

p. 147 # 41 – 49 odd
p. 518 # 1 – 9 odd