7.1 Integral as Net Change
A honey bee makes several trips from the hive to a flower garden. The velocity graph is shown below.

What is the total distance traveled by the bee?

200 ft + 200 ft + 200 ft + 100 ft = 700 ft
What is the displacement of the bee?

\[200 - 200 + 200 - 100 = 100\]

100 feet towards the hive
To find the displacement (position shift) from the velocity function, we just integrate the function. The negative areas below the x-axis subtract from the total displacement.

\[ \text{Displacement} = \int_{t_1}^{t_2} v(t) \, dt \]

To find distance traveled we have to use absolute value.

\[ \text{Distance Traveled} = \int_{t_1}^{t_2} |v(t)| \, dt \]

Find the roots of the velocity equation and integrate in pieces, just like when we found the area between a curve and the x-axis. (Take the absolute value of each integral.)

Or you can use your calculator to integrate the absolute value of the velocity function.
Recall for Rectilinear Motion (motion along a line):

- **Position function** = \( x(t) \)
- **Velocity function** = \( v(t) = x'(t) \)

\[\begin{align*}
v(t) > 0 & \implies \text{moving in positive direction} \\
v(t) < 0 & \implies \text{moving in negative direction} \\
v(t) = 0 & \implies \text{object stopped/changing direction}
\end{align*}\]

\[|v(t)| = \text{Speed}\]

- **Acceleration function** = \( a(t) = v'(t) = x''(t) \)

- If \( a(t) \) and \( v(t) \) have the same sign \( \implies \text{speeding up} \)
- If \( a(t) \) and \( v(t) \) have opp. signs \( \implies \text{slowing down} \)
Displacement: \( \Delta x = x(t_2) - x(t_1) \)

From integration:
\[
\int_{t_1}^{t_2} v(t) \, dt = x(t)|_{t_1}^{t_2} = x(t_2) - x(t_1) = \Delta x
\]

A variation of the displacement formula:
\[
x(t_2) = x(t_1) + \int_{t_1}^{t_2} v(t) \, dt
\]

Final pos. = initial position + displacement
Total Distance Traveled: must account for positive and negative velocities

\[ \text{Total Distance} = \int_{t_1}^{t_2} |v(t)| \, dt \]
Example: Suppose that \[ v(t) = t^2 - \frac{8}{(t + 1)^2} \text{ cm/s} \]

a. Describe the motion of the object for the first five seconds. Support your answer graphically.

\[ v(t) = 0 \text{ at } t = 1.2545 \]

\[ v(t) > 0 \text{ on } (1.2545, \infty) \Rightarrow \text{object moving right} \]

\[ v(t) < 0 \text{ on } (0, 1.2545) \Rightarrow \text{object moves left} \]

\[ v(t) \text{ increases on } (0, 5) \Rightarrow a(t) > 0 \text{ on } (0, 5), \text{ object speeding up} \]

\[ v(t) \text{ decreases on } (1.2545, 5) \text{ since } v(t) > 0 \text{ and } a(t) > 0. \]

\[ \text{Object slowing down on } (0, 1.2545) \text{ since } v(t) \text{ and } a(t) \text{ are opposite signs.} \]
Example: Suppose that \( v(t) = t^2 - \frac{8}{(t + 1)^2} \) cm/s

b. If the initial position of the object is 9, what is the position of the object at 5 seconds?
c. How far did the object travel over the first five seconds?

\[
b) \quad x(s) = x(0) + \int_0^5 v(t) \, dt = 9 + \int_0^5 v(t) \, dt = 44 \text{ cm}
\]

\[
c) \quad \int_0^5 |v(t)| \, dt = 42.587 \text{ cm}
\]
Example: Suppose that a car with initial velocity 5 mph has \( a(t) = 2.4t \) mph/s.

How fast is the car going at the end of 8 seconds?

\[
v(8) = v(0) + \int_0^8 a(t) \, dt
\]

\[
= 5 + \left[ 1.2t^2 \right]_0^8
\]

\[
= 5 + 1.2(64)
\]

\[
= 5 + 76.8 = 81.8 \text{ mph}
\]
2. A particle moves along the x-axis with velocity given by \( v(t) = \frac{10 \sin(0.4t^2)}{t^2 - t + 3} \) for time \( 0 \leq t \leq 3.5 \).

The particle is at position \( x = -5 \) at time \( t = 0 \).

(a) Find the acceleration of the particle at time \( t = 3 \).

\[ a(3) = v'(3) = -2.118 \]
2. A particle moves along the $x$-axis with velocity given by $v(t) = \frac{10 \sin(0.4t^2)}{t^2 - t + 3}$ for time $0 \leq t \leq 3.5$.

The particle is at position $x = -5$ at time $t = 0$.

(b) Find the position of the particle at time $t = 3$.

$$x(3) = -5 + \int_{0}^{3} v(t) \, dt = -5 + \int_{0}^{3} \frac{10 \sin(0.4t^2)}{t^2 - t + 3} \, dt = -1.76025478113501 \text{ m}$$
2. A particle moves along the \( x \)-axis with velocity given by \( v(t) = \frac{10 \sin (0.4t^2)}{t^2 - t + 3} \) for time \( 0 \leq t \leq 3.5 \).

The particle is at position \( x = -5 \) at time \( t = 0 \).

(c) Evaluate \( \int_{0}^{3.5} v(t) \, dt \), and evaluate \( \int_{0}^{3.5} |v(t)| \, dt \). Interpret the meaning of each integral in the context of the problem.

\[ \int_{0}^{3.5} v(t) \, dt = 2.844 \quad \text{THE DISPLACEMENT OF PARTICLE FROM } t=0 \text{ TO } t=3.5 \]

\[ \int_{0}^{3.5} |v(t)| \, dt = 3.737 \quad \text{TOTAL DIST. TRAVELED BY PARTICLE FROM } t=0 \text{ TO } t=3.5 \]
2. A particle moves along the $x$-axis with velocity given by $v(t) = \frac{10 \sin(0.4t^2)}{t^2 - t + 3}$ for time $0 \leq t \leq 3.5$.

The particle is at position $x = -5$ at time $t = 0$.

(d) A second particle moves along the $x$-axis with position given by $x_2(t) = t^2 - t$ for $0 \leq t \leq 3.5$. At what time $t$ are the two particles moving with the same velocity?

$x_2'(t) = 2t - 1 = v(t) \Rightarrow t = 1.570$
Classwork:
Chapter 7 AP Packet #2, 4, 6, 10, 18

Homework:
Chapter 7 AP Packet #1, 3, 5, 13, 17, 21
Example: National Potato Consumption

The rate of potato consumption for a particular country was:

\[ C(t) = 2.2 + 1.1t \]

where \( t \) is the number of years since 1970 and \( C \) is in millions of bushels per year.

Find the amount of potatoes consumed from the beginning of 1972 to the end of 1973.
Example 5: National Potato Consumption

\[ C(t) = 2.2 + 1.1^t \]

To find the cumulative effect over time – Integrate it!

From the beginning of 1972 to the end of 1973:

\[
\int_{2}^{4} (2.2 + 1.1^t) \, dt = 2.2t + \frac{1}{\ln 1.1} 1.1^t \bigg|_{2}^{4} \approx 7.066 \text{ million bushels}
\]
Net Change from Data: what if we don’t have the a function to work with?

Example (p. 369): A pump connected to a generator operates at a varying rate, depending on how much power is being drawn from the generator. The rate (gallons per minute) at which the pump operates is recorded at 5-minute intervals for an hour as shown in the table. How many gallons were pumped during the hour?

<table>
<thead>
<tr>
<th>Time (min)</th>
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Gallons pumped = \[ \int_{0}^{60} R(t) dt \]

We don’t have a formula for \( R(t) \), so we have to approximate the integral – the trapezoidal rule works well:

\[
\frac{1}{2} \cdot 5 \cdot [58 + 2(60) + \cdots + 2(63) + 63] \\
= 3582.5 \text{ gallons}
\]
2010 BC1

There is no snow on Janet’s driveway when snow begins to fall at midnight. From midnight to 9 A.M., snow accumulates on the driveway at a rate modeled by $f(t) = 7te^{\cos t}$ cubic feet per hour, where $t$ is measured in hours since midnight. Janet starts removing snow at 6 A.M. ($t = 6$). The rate $g(t)$, in cubic feet per hour, at which Janet removes snow from the driveway at time $t$ hours after midnight is modeled by

$$g(t) = \begin{cases} 
0 & \text{for } 0 \leq t < 6 \\
125 & \text{for } 6 \leq t < 7 \\
108 & \text{for } 7 \leq t \leq 9.
\end{cases}$$

(a) How many cubic feet of snow have accumulated on the driveway by 6 A.M.?
2010 BC1

There is no snow on Janet’s driveway when snow begins to fall at midnight. From midnight to 9 A.M., snow accumulates on the driveway at a rate modeled by \( f(t) = 7te^{\cos t} \) cubic feet per hour, where \( t \) is measured in hours since midnight. Janet starts removing snow at 6 A.M. \( (t = 6) \). The rate \( g(t) \), in cubic feet per hour, at which Janet removes snow from the driveway at time \( t \) hours after midnight is modeled by

\[
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0 & \text{for } 0 \leq t < 6 \\
125 & \text{for } 6 \leq t < 7 \\
108 & \text{for } 7 \leq t \leq 9.
\end{cases}
\]

(b) Find the rate of change of the volume of snow on the driveway at 8 A.M.
There is no snow on Janet’s driveway when snow begins to fall at midnight. From midnight to 9 A.M., snow accumulates on the driveway at a rate modeled by \( f(t) = 7te^{\cos t} \) cubic feet per hour, where \( t \) is measured in hours since midnight. Janet starts removing snow at 6 A.M. \((t = 6)\). The rate \( g(t) \), in cubic feet per hour, at which Janet removes snow from the driveway at time \( t \) hours after midnight is modeled by

\[
g(t) = \begin{cases} 
0 & \text{for } 0 \leq t < 6 \\
125 & \text{for } 6 \leq t < 7 \\
108 & \text{for } 7 \leq t \leq 9.
\end{cases}
\]

Let \( h(t) \) represent the total amount of snow, in cubic feet, that Janet has removed from the driveway at time \( t \) hours after midnight. Express \( h \) as a piecewise-defined function with domain \( 0 \leq t \leq 9 \).
2010 BC1

There is no snow on Janet’s driveway when snow begins to fall at midnight. From midnight to 9 A.M., snow accumulates on the driveway at a rate modeled by $f(t) = 7te^{\cos t}$ cubic feet per hour, where $t$ is measured in hours since midnight. Janet starts removing snow at 6 A.M. ($t = 6$). The rate $g(t)$, in cubic feet per hour, at which Janet removes snow from the driveway at time $t$ hours after midnight is modeled by

$$g(t) = \begin{cases} 0 & \text{for } 0 \leq t < 6 \\ 125 & \text{for } 6 \leq t < 7 \\ 108 & \text{for } 7 \leq t \leq 9. \end{cases}$$

(d) How many cubic feet of snow are on the driveway at 9 A.M.?
Classwork:

Chapter 7 AP Packet #23 – 25

Homework:

Chapter 7 AP Packet #27 – 29