This print-out should have 9 questions. Multiple-choice questions may continue on the next column or page – find all choices before answering.

BC 1993 FR 4
001 (part 1 of 2) 10.0 points
Consider the polar curve \( r = 2 \sin(3\theta) \) for \( 0 \leq \theta \leq \pi \).

a) Find the area of region inside the curve.

\[ A = \frac{1}{2} \int_{0}^{\pi} 4 \sin^2 3\theta \, d\theta \]

\[ = \left. \theta - \frac{1}{6} \sin 6\theta \right|_{0}^{\pi} = \pi \]

Alternate Solution 1:
\[ A = \frac{3}{2} \int_{0}^{\pi} 4 \sin^2 3\theta \, d\theta = \cdots = \pi \]

Alternate Solution 2:
\[ A = \frac{6}{2} \int_{0}^{\frac{\pi}{2}} 4 \sin^2 3\theta \, d\theta = \cdots = \pi \]

1. \( \frac{1}{4} \)
2. \( \frac{1}{2} \) correct
3. \( \frac{1}{5} \)
4. \( \frac{1}{3} \)
5. \( 0 \)

Explanation:

\[ x = 2 \sin 3\theta \cos \theta \]
\[ \frac{dx}{d\theta} = -2 \sin 3\theta \sin \theta + 6 \cos 3\theta \cos \theta \]
\[ y = 2 \sin 3\theta \sin \theta \]
\[ \frac{dy}{d\theta} = 2 \sin 3\theta \cos \theta + 6 \cos 3\theta \sin \theta \]

At \( \theta = \frac{\pi}{4} \), \( \frac{dy}{d\theta} = -2 \) and \( \frac{dx}{d\theta} = -4 \), so
\[ \frac{dy}{dx} = -\frac{2}{-4} = \frac{1}{2} \]

Alternate Solution:
\[ (x^2 + y^2)^2 = 6x^2y - 2y^3 \]
\[ 2 (x^2 + y^2) \left( 2x + 2y \frac{dy}{dx} \right) = 6x^2 \frac{dy}{dx} - 6y^2 \frac{dy}{dx} + 12xy \]

At \( \theta = \frac{\pi}{4} \), \( x = 1 \) and \( y = 1 \), so
\[ 4 \left( 2 + 2 \frac{dy}{dx} \right) = 6 \frac{dy}{dx} + 12 - 6 \frac{dy}{dx} \]
\[ 8 + 8 \frac{dy}{dx} = 12 \Rightarrow \frac{dy}{dx} = \frac{1}{2} \]

BC 1997 MCnc 15
003 10.0 points
The length of the path described by the parametric equations \( x = \cos^3 t \) and \( y = \sin^3 t \), for \( 0 \leq t \leq \frac{\pi}{2} \), is given by

1. \( \int_{0}^{\frac{\pi}{2}} \sqrt{9 \cos^4 t \sin^2 t + 9 \sin^4 t \cos^2 t} \, dt \) correct
2. $\int_0^{\pi/2} \sqrt{-3 \cos^2 t \sin t + 3 \sin^2 t \cos t} \, dt$

3. $\int_0^{\pi/2} \sqrt{3 \cos^2 t + 3 \sin^2 t} \, dt$

4. $\int_0^{\pi/2} \sqrt{9 \cos^4 t + 9 \sin^4 t} \, dt$

5. $\int_0^{\pi/2} \sqrt{\cos^6 t + \sin^6 t} \, dt$

**Explanation:**

**Basic Concept:**

Path length is

$$L = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt$$

**Solution:**

$$x = \cos^3 t$$

$$\frac{dx}{dt} = 3 \cos^2 t \, (-\sin t)$$

$$y = \sin^3 t$$

$$\frac{dy}{dt} = 3 \sin^2 t \, (\cos t)$$

$$L = \int_0^{\pi/2} \sqrt{9 \cos^4 t \sin^2 t + 9 \sin^4 t \cos^2 t} \, dt$$

Because of symmetry with respect to the $x$-axis,

$$A = \frac{1}{2} \int_0^{\pi/2} 4 \cos^2 \theta \, d\theta - \frac{1}{2} \int_0^{\pi/2} \cos^2 \theta \, d\theta$$

$$A = 3 \int_0^{\pi/2} \cos^2 \theta \, d\theta$$

The shaded region in

lies inside the polar curve $r = 3 \sin \theta$ and outside the polar curve $r = 2 \cos \theta$. Which of the following integrals gives the area of this region?

1. $I = \int_0^{\pi} 5 \sin^2 \theta \, d\theta$

2. $I = \frac{1}{2} \int_0^{\pi/2} 5 \sin^2 \theta \, d\theta$ **correct**

3. $I = \int_0^{\pi} \sin \theta \, d\theta$

4. $I = \frac{1}{2} \int_0^{\pi} 5 \sin^2 \theta \, d\theta$ **correct**

5. $I = \frac{1}{2} \int_0^{\pi} \sin \theta \, d\theta$
6. \[ I = \frac{1}{2} \int_{0}^{\pi/2} \sin \theta \, d\theta \]

**Explanation:**
As the graphs show, the polar curves intersect when
\[ 3 \sin \theta = 2 \sin \theta , \]
*i.e.* at \( \theta = 0, \pi \). Thus the area of the shaded region is given by
\[ I = \frac{1}{2} \int_{0}^{\pi} \left\{ (3 \sin \theta)^2 - (2 \sin \theta)^2 \right\} \, d\theta . \]
Consequently,
\[ I = \frac{1}{2} \int_{0}^{\pi} 5 \sin^2 \theta \, d\theta . \]

**keywords:** area, polar coordinates, definite integral, circle,

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**CalC11d06a**

006 10.0 points

Find the area of the region enclosed by the graph of the polar function
\[ r = 4 + \sin \theta . \]

1. area = \( \frac{35}{2} \pi \)
2. area = \( 18 \pi \)
3. area = \( \frac{33}{2} \pi \) **correct**
4. area = \( 17 \pi \)
5. area = \( 35 \pi \)

**Explanation:**
The area of a region bounded by the graph of the polar function \( r = f(\theta) \) and the rays \( \theta = \theta_0, \theta_1 \) is given by the integral
\[ A = \frac{1}{2} \int_{\theta_0}^{\theta_1} f(\theta)^2 \, d\theta . \]

On the other hand, the graph of
\[ r = 4 + \sin \theta \]
is the cardioid similar to the one shown in

so in this case we can take \( \theta_0 = 0 \) and \( \theta_1 = 2\pi \). Thus the area of the region enclosed by the graph is given by the integral
\[ A = \frac{1}{2} \int_{0}^{2\pi} (4 + \sin \theta)^2 \, d\theta . \]

Now
\[ (4 + \sin \theta)^2 = 16 + 2 \sin \theta + \sin^2 \theta \]
\[ = \frac{33}{2} + 2 \sin \theta - \frac{1}{2} \cos 2\theta , \]
since
\[ \sin^2 \theta = \frac{1}{2} \left( 1 - \cos 2\theta \right) . \]

But then,
\[ A = \frac{1}{2} \int_{0}^{2\pi} \left( \frac{33}{2} + 2 \sin \theta - \frac{1}{2} \cos 2\theta \right) \, d\theta \]
\[ = \frac{1}{2} \left[ \frac{33}{2} \theta - 2 \cos \theta - \frac{1}{4} \sin 2\theta \right]_{0}^{2\pi} . \]

Consequently,
\[ \text{area} = A = \frac{33}{2} \pi . \]

**keywords:** polar graph, area, cardioid

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**CalC11d07a**
Find the area of one loop of the graph of the polar function

\[ r = 3 \cos 4\theta. \]

1. area = \( \frac{9}{16} \pi \) correct
2. area = \( \frac{21}{32} \pi \)
3. area = \( \frac{19}{32} \pi \)
4. area = \( \frac{11}{16} \pi \)
5. area = \( \frac{5}{8} \pi \)

**Explanation:**
The area of a region bounded by the graph of the polar function \( r = f(\theta) \) and the rays \( \theta = \theta_0, \theta_1 \) is given by the integral

\[ A = \frac{1}{2} \int_{\theta_0}^{\theta_1} f(\theta)^2 \, d\theta. \]

Now the graph of

\[ r = 3 \cos 4\theta \]

is the ‘rose’

\[ \theta_0 = \frac{-\pi}{8}, \quad \theta_1 = \frac{\pi}{8}. \]

Thus

\[ A = \frac{1}{2} \int_{-\pi/8}^{\pi/8} (3 \cos 4\theta)^2 \, d\theta \]

\[ = \frac{9}{4} \int_{-\pi/8}^{\pi/8} (1 + \cos 8\theta) \, d\theta \]

\[ = \frac{9}{4} \left[ \theta + \frac{1}{8} \sin 8\theta \right]_{-\pi/8}^{\pi/8}. \]

Consequently,

\[ \text{area} = A = \frac{9}{16} \pi. \]

**keywords:** polar graph, area, cardioid,

CalC11d11a

Find the area of the shaded region in

specified by the graphs of the circles

\[ r = \cos \theta, \quad r = \sin \theta. \]

1. area = \( \frac{1}{4} \)
2. area = \( \frac{\pi}{4} \)
3. area = \( \frac{\pi}{2} \)
4. area = \( \frac{1}{4} \left( \frac{\pi}{2} - 1 \right) \)
5. area $= \frac{1}{2} \left( \frac{\pi}{4} + 1 \right)$

6. area $= \frac{1}{4} \left( \frac{\pi}{2} + 1 \right)$ correct

Explanation:
The area of a region bounded by the graph of polar function $r = f(\theta)$ between the rays $\theta = \theta_0, \theta_1$ is given by the integral

$$A = \frac{1}{2} \int_{\theta_0}^{\theta_1} f(\theta)^2 \, d\theta.$$ 

Now the graph of $r = \cos \theta$ is a circle centered on the $x$-axis, while that of $r = \sin \theta$ is a circle centered on the $y$-axis; in addition, both pass through the origin and have the same radius. So the circles intersect also at $\theta = \pi/4$, as the figure shows.

The area of the shaded region can thus be computed as the difference

$$I_1 - I_2 = \frac{1}{2} \int_{\pi/4}^{\pi} \sin^2 \theta \, d\theta - \frac{1}{2} \int_{\pi/4}^{\pi/2} \cos^2 \theta \, d\theta$$

of the area of the respective shaded regions in

Consequently, the shaded region has

$$\text{area } = \frac{1}{4} \left( \frac{\pi}{2} + 1 \right).$$

keywords: definite integral, area between curves, polar area, circle,

CalC11d16a

009 10.0 points

Find the area of the region lying outside the polar curve

$$r = 2(1 + \sin \theta)$$

and inside the polar curve $r = 6 \sin \theta$.

1. area $= 2\pi$

2. area $= 4\pi$ correct

3. area $= \pi$

4. area $= \frac{1}{2} \pi$

5. area $= 3\pi$

Explanation:
The area of a region bounded by the graph of the polar function $r = f(\theta)$ and the rays $\theta = \theta_0, \theta_1$ is given by the integral

$$A = \frac{1}{2} \int_{\theta_0}^{\theta_1} f(\theta)^2 \, d\theta.$$ 

the graph of

$$r = 2(1 + \sin \theta)$$

is a cardioid, while the graph of $r = 6 \sin \theta$ is a circle passing through the origin, having center on the $y$-axis and diameter 6, so the region between them is the shaded crescent in
The values of $\theta_0$, $\theta_1$ will be the solutions of

$$2(1 + \sin \theta) = 6 \sin \theta,$$

i.e., $\theta_0 = \pi/6$, $\theta_1 = 5\pi/6$, as the graph confirms. Thus the area of the shaded region is given by

$$A = \frac{1}{2} \int_{\pi/6}^{5\pi/6} \left\{ (6 \sin \theta)^2 - 4(1 + \sin \theta)^2 \right\} d\theta.$$

Now

$$(6 \sin \theta)^2 - 4(1 + \sin \theta)^2$$

$$= 4(9 \sin^2 \theta - (1 + 2 \sin \theta + \sin^2 \theta))$$

$$= 4(8 \sin^2 \theta - 2 \sin \theta - 1)$$

$$= 4(3 - 2 \sin \theta - 4 \cos 2\theta).$$

But then

$$A = 2 \int_{\pi/6}^{5\pi/6} \left\{ 3 - 2 \sin \theta - 4 \cos 2\theta \right\} d\theta$$

$$= 2 \left[ 3\theta + 2 \cos \theta - 2 \sin 2\theta \right]_{\pi/6}^{5\pi/6}.$$

Consequently,

$$\text{area} = A = 4\pi.$$

keywords: area, polar curves, polar graphs, circle, cardioid