# 2.2 Polynomial Functions of Higher Degree

### What you should learn

- Use transformations to sketch graphs of polynomial functions.
- Use the Leading Coefficient Test to determine the end behavior of graphs of polynomial functions.
- Find and use zeros of polynomial functions as sketching aids.
- Use the Intermediate Value Theorem to help locate zeros of polynomial functions.

### Why you should learn it

You can use polynomial functions to analyze business situations such as how revenue is related to advertising expenses, as discussed in Exercise 98 on page 151.


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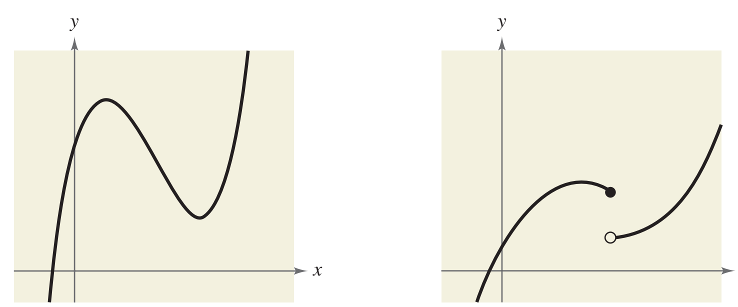
## Graphs of Polynomial Functions

In this section, you will study basic features of the graphs of polynomial functions. The first feature is that the graph of a polynomial function is **continuous.** Essentially, this means that the graph of a polynomial function has no breaks, holes, or gaps, as shown in Figure 2.10(a). The graph shown in Figure 2.10(b) is an example of a piecewise-defined function that is not continuous.

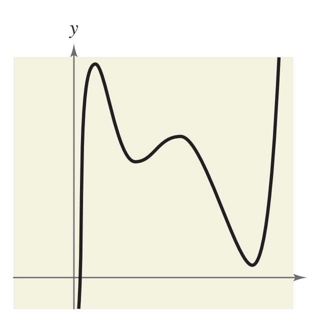



**(a) Polynomial functions have continuous graphs.**

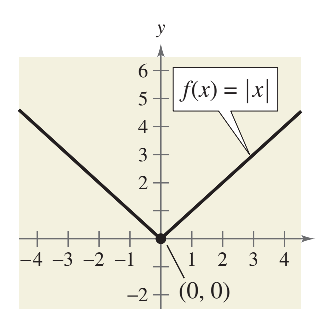
**(b) Functions with graphs that are not continuous are not polynomial functions.**

FIGURE 2.10

The second feature is that the graph of a polynomial function has only smooth, rounded turns, as shown in Figure 2.11. A polynomial function cannot have a sharp turn. For instance, the function given by $f(x) = |x|$, which has a sharp turn at the point $(0, 0)$, as shown in Figure 2.12, is not a polynomial function.




*Polynomial functions have graphs with smooth rounded turns.*
FIGURE 2.11




*Graphs of polynomial functions cannot have sharp turns.*
FIGURE 2.12

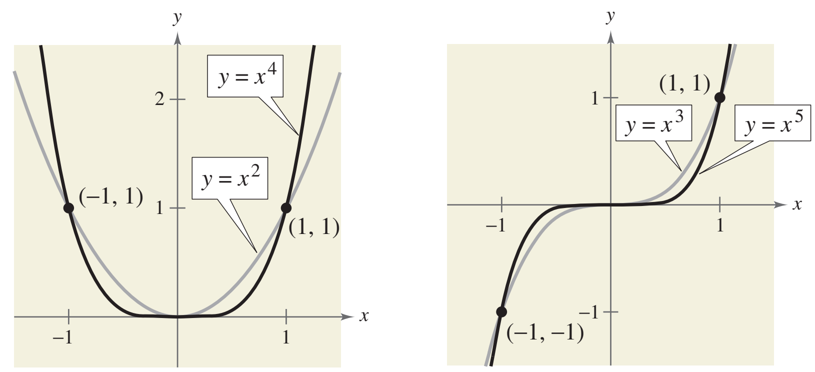
Point out to students that although all polynomial functions are continuous and have rounded turns, not all graphs that are continuous and have rounded turns are polynomials.

The graphs of polynomial functions of degree greater than 2 are more difficult to analyze than the graphs of polynomials of degree 0, 1, or 2. However, using the features presented in this section, coupled with your knowledge of point plotting, intercepts, and symmetry, you should be able to make reasonably accurate sketches *by hand*.

**STUDY TIP**

For power functions given by $f(x) = x^n$, if $n$ is even, then the graph of the function is symmetric with respect to the $y$-axis, and if $n$ is odd, then the graph of the function is symmetric with respect to the origin.

The polynomial functions that have the simplest graphs are monomials of the form $f(x) = x^n$, where $n$ is an integer greater than zero. From Figure 2.13, you can see that when $n$ is *even*, the graph is similar to the graph of $f(x) = x^2$, and when $n$ is *odd*, the graph is similar to the graph of $f(x) = x^3$. Moreover, the greater the value of $n$, the flatter the graph near the origin. Polynomial functions of the form $f(x) = x^n$ are often referred to as **power functions.**




(a) If $n$ is even, the graph of $y = x^n$ touches the axis at the $x$-intercept.

(b) If $n$ is odd, the graph of $y = x^n$ crosses the axis at the $x$-intercept.

FIGURE 2.13

## Example 1 Sketching Transformations of Monomial Functions

Sketch the graph of each function.

**a.** $f(x) = -x^5$ **b.** $h(x) = (x + 1)^4$

### Solution

**a.** Because the degree of $f(x) = -x^5$ is odd, its graph is similar to the graph of $y = x^3$. In Figure 2.14, note that the negative coefficient has the effect of reflecting the graph in the $x$-axis.

**b.** The graph of $h(x) = (x + 1)^4$, as shown in Figure 2.15, is a left shift by one unit of the graph of $y = x^4$.

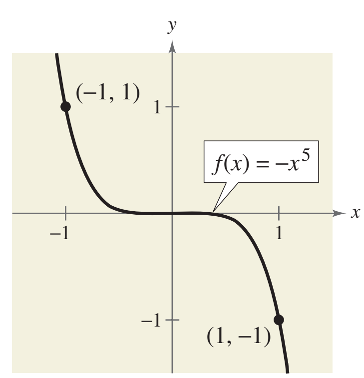



FIGURE 2.14

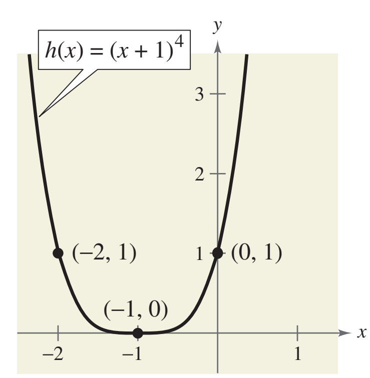



FIGURE 2.15

CHECKPOINT Now try Exercise 9.

## Exploration

For each function, identify the degree of the function and whether the degree of the function is even or odd. Identify the leading coefficient and whether the leading coefficient is positive or negative. Use a graphing utility to graph each function. Describe the relationship between the degree and the sign of the leading coefficient of the function and the right-hand and left-hand behavior of the graph of the function.

**a.** $f(x) = x^3 - 2x^2 - x + 1$
**b.** $f(x) = 2x^5 + 2x^2 - 5x + 1$
**c.** $f(x) = -2x^5 - x^2 + 5x + 3$
**d.** $f(x) = -x^3 + 5x - 2$
**e.** $f(x) = 2x^2 + 3x - 4$
**f.** $f(x) = x^4 - 3x^2 + 2x - 1$
**g.** $f(x) = x^2 + 3x + 2$

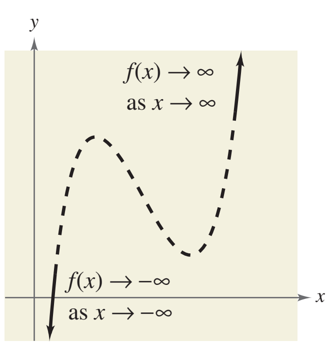
## The Leading Coefficient Test

In Example 1, note that both graphs eventually rise or fall without bound as $x$ moves to the right. Whether the graph of a polynomial function eventually rises or falls can be determined by the function's degree (even or odd) and by its leading coefficient, as indicated in the **Leading Coefficient Test.**

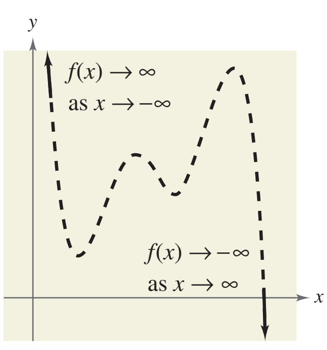
### Leading Coefficient Test

As $x$ moves without bound to the left or to the right, the graph of the polynomial function $f(x) = a_n x^n + \cdot \cdot \cdot + a_1 x + a_0$ eventually rises or falls in the following manner.

**1.** When $n$ is *odd:*

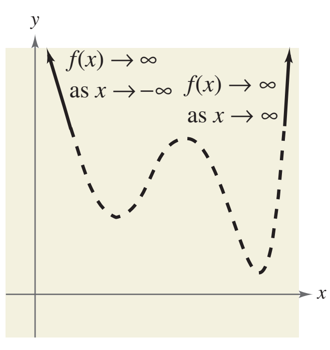



If the leading coefficient is positive $(a_n > 0)$, the graph falls to the left and rises to the right.

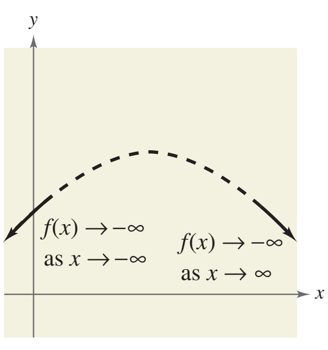



If the leading coefficient is negative $(a_n < 0)$, the graph rises to the left and falls to the right.

**2.** When $n$ is *even:*




If the leading coefficient is positive $(a_n > 0)$, the graph rises to the left and right.




If the leading coefficient is negative $(a_n < 0)$, the graph falls to the left and right.

The dashed portions of the graphs indicate that the test determines *only* the right-hand and left-hand behavior of the graph.

## STUDY TIP

The notation "$f(x) \to -\infty$ as $x \to -\infty$" indicates that the graph falls to the left. The notation "$f(x) \to \infty$ as $x \to \infty$" indicates that the graph rises to the right.

A review of the shapes of the graphs of polynomial functions of degrees 0, 1, and 2 may be used to illustrate the Leading Coefficient Test.

**STUDY TIP**

A polynomial function is written in **standard form** if its terms are written in descending order of exponents from left to right. Before applying the Leading Coefficient Test to a polynomial function, it is a good idea to check that the polynomial function is written in standard form.

### Example 2 Applying the Leading Coefficient Test

Describe the right-hand and left-hand behavior of the graph of each function.

**a.** $f(x) = -x^3 + 4x$ **b.** $f(x) = x^4 - 5x^2 + 4$ **c.** $f(x) = x^5 - x$

**Solution**

**a.** Because the degree is odd and the leading coefficient is negative, the graph rises to the left and falls to the right, as shown in Figure 2.16.

**b.** Because the degree is even and the leading coefficient is positive, the graph rises to the left and right, as shown in Figure 2.17.

**c.** Because the degree is odd and the leading coefficient is positive, the graph falls to the left and rises to the right, as shown in Figure 2.18.

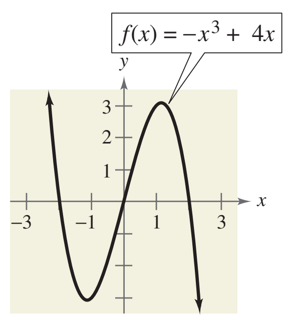



FIGURE 2.16

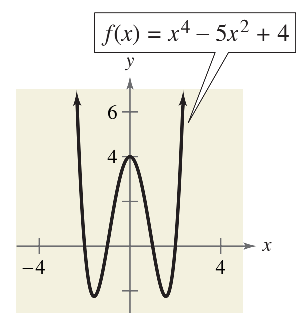



FIGURE 2.17

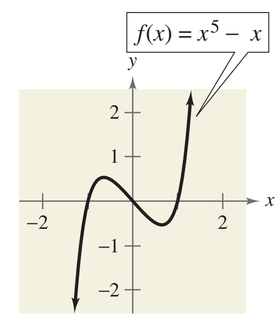



FIGURE 2.18

**Exploration**

For each of the graphs in Example 2, count the number of zeros of the polynomial function and the number of relative minima and relative maxima. Compare these numbers with the degree of the polynomial. What do you observe?

CHECKPOINT Now try Exercise 15.

In Example 2, note that the Leading Coefficient Test tells you only whether the graph *eventually* rises or falls to the right or left. Other characteristics of the graph, such as intercepts and minimum and maximum points, must be determined by other tests.

## Zeros of Polynomial Functions

It can be shown that for a polynomial function $f$ of degree $n$, the following statements are true.

1. The function $f$ has, at most, $n$ real zeros. (You will study this result in detail in the discussion of the Fundamental Theorem of Algebra in Section 2.5.)
2. The graph of $f$ has, at most, $n - 1$ turning points. (Turning points, also called relative minima or relative maxima, are points at which the graph changes from increasing to decreasing or vice versa.)

**STUDY TIP**

Remember that the *zeros* of a function of $x$ are the $x$-values for which the function is zero.

Finding the zeros of polynomial functions is one of the most important problems in algebra. There is a strong interplay between graphical and algebraic approaches to this problem. Sometimes you can use information about the graph of a function to help find its zeros, and in other cases you can use information about the zeros of a function to help sketch its graph. Finding zeros of polynomial functions is closely related to factoring and finding $x$-intercepts.

### Real Zeros of Polynomial Functions

If $f$ is a polynomial function and $a$ is a real number, the following statements are equivalent.

**1.** $x = a$ is a *zero* of the function $f$.

**2.** $x = a$ is a *solution* of the polynomial equation $f(x) = 0$.

**3.** $(x - a)$ is a *factor* of the polynomial $f(x)$.

**4.** $(a, 0)$ is an *x-intercept* of the graph of $f$.

## Example 3 Finding the Zeros of a Polynomial Function

Find all real zeros of

$$f(x) = -2x^4 + 2x^2.$$

Then determine the number of turning points of the graph of the function.

### Algebraic Solution

To find the real zeros of the function, set $f(x)$ equal to zero and solve for $x$.

$$-2x^4 + 2x^2 = 0 \quad \text{Set } f(x) \text{ equal to 0.}$$

$$-2x^2(x^2 - 1) = 0 \quad \text{Remove common monomial factor.}$$

$$-2x^2(x - 1)(x + 1) = 0 \quad \text{Factor completely.}$$

So, the real zeros are $x = 0$, $x = 1$, and $x = -1$. Because the function is a fourth-degree polynomial, the graph of $f$ can have at most $4 - 1 = 3$ turning points.

CHECKPOINT Now try Exercise 27.

### Graphical Solution

Use a graphing utility to graph $y = -2x^4 + 2x^2$. In Figure 2.19, the graph appears to have zeros at $(0, 0)$, $(1, 0)$, and $(-1, 0)$. Use the *zero* or *root* feature, or the *zoom* and *trace* features, of the graphing utility to verify these zeros. So, the real zeros are $x = 0$, $x = 1$, and $x = -1$. From the figure, you can see that the graph has three turning points. This is consistent with the fact that a fourth-degree polynomial can have at most three turning points.

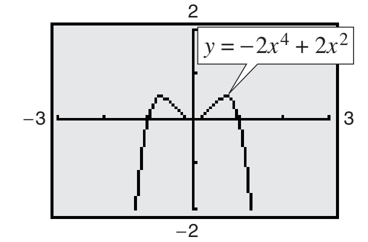



FIGURE 2.19

In Example 3, note that because $k$ is even, the factor $-2x^2$ yields the *repeated* zero $x = 0$. The graph touches the $x$-axis at $x = 0$, as shown in Figure 2.19.

### Repeated Zeros

A factor $(x - a)^k$, $k > 1$, yields a **repeated zero** $x = a$ of **multiplicity** $k$.

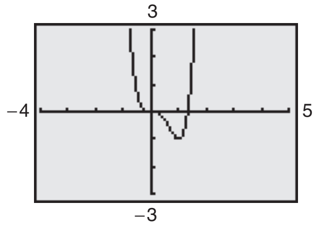
**1.** If $k$ is odd, the graph *crosses* the $x$-axis at $x = a$.

**2.** If $k$ is even, the graph *touches* the $x$-axis (but does not cross the $x$-axis) at $x = a$.

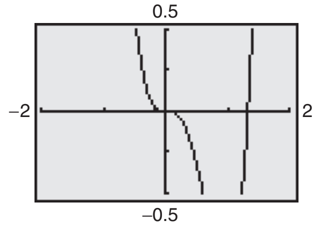
## Technology

Example 4 uses an *algebraic approach* to describe the graph of the function. A graphing utility is a complement to this approach. Remember that an important aspect of using a graphing utility is to find a viewing window that shows all significant features of the graph. For instance, the viewing window in part (a) illustrates all of the significant features of the function in Example 4.

**a.**




**b.**




To graph polynomial functions, you can use the fact that a polynomial function can change signs only at its zeros. Between two consecutive zeros, a polynomial must be entirely positive or entirely negative. This means that when the real zeros of a polynomial function are put in order, they divide the real number line into intervals in which the function has no sign changes. These resulting intervals are **test intervals** in which a representative $x$-value in the interval is chosen to determine if the value of the polynomial function is positive (the graph lies above the $x$-axis) or negative (the graph lies below the $x$-axis).

### Example 4 Sketching the Graph of a Polynomial Function

Sketch the graph of $f(x) = 3x^4 - 4x^3$.

**Solution**

1. *Apply the Leading Coefficient Test.* Because the leading coefficient is positive and the degree is even, you know that the graph eventually rises to the left and to the right (see Figure 2.20).
2. *Find the Zeros of the Polynomial.* By factoring $f(x) = 3x^4 - 4x^3$ as $f(x) = x^3(3x - 4)$, you can see that the zeros of $f$ are $x = 0$ and $x = \frac{4}{3}$ (both of odd multiplicity). So, the $x$-intercepts occur at $(0, 0)$ and $\left(\frac{4}{3}, 0\right)$. Add these points to your graph, as shown in Figure 2.20.
3. *Plot a Few Additional Points.* Use the zeros of the polynomial to find the test intervals. In each test interval, choose a representative $x$-value and evaluate the polynomial function, as shown in the table.

| Test interval | Representative $x$-value | Value of $f$ | Sign | Point on graph |
|---|---|---|---|---|
| $(-\infty, 0)$ | $-1$ | $f(-1) = 7$ | Positive | $(-1, 7)$ |
| $\left(0, \frac{4}{3}\right)$ | $1$ | $f(1) = -1$ | Negative | $(1, -1)$ |
| $\left(\frac{4}{3}, \infty\right)$ | $1.5$ | $f(1.5) = 1.6875$ | Positive | $(1.5, 1.6875)$ |

4. *Draw the Graph.* Draw a continuous curve through the points, as shown in Figure 2.21. Because both zeros are of odd multiplicity, you know that the graph should cross the $x$-axis at $x = 0$ and $x = \frac{4}{3}$.

## STUDY TIP

If you are unsure of the shape of a portion of the graph of a polynomial function, plot some additional points, such as the point $(0.5, -0.3125)$ as shown in Figure 2.21.

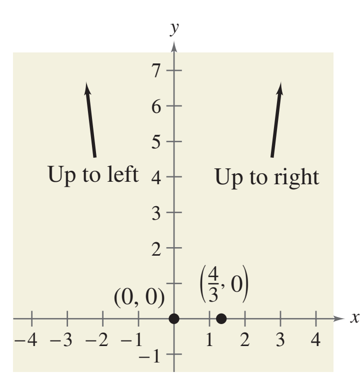



FIGURE 2.20

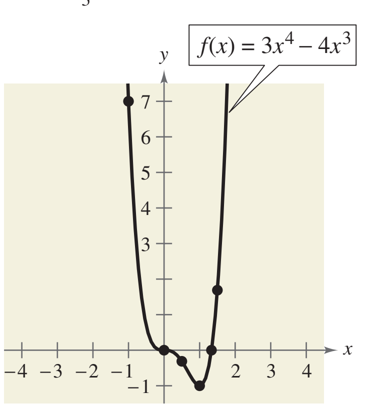



FIGURE 2.21

CHECKPOINT Now try Exercise 67.

## Example 5 Sketching the Graph of a Polynomial Function

Sketch the graph of $f(x) = -2x^3 + 6x^2 - \frac{9}{2}x$.

### Solution

1. *Apply the Leading Coefficient Test.* Because the leading coefficient is negative and the degree is odd, you know that the graph eventually rises to the left and falls to the right (see Figure 2.22).
2. *Find the Zeros of the Polynomial.* By factoring

$$\begin{aligned} f(x) &= -2x^3 + 6x^2 - \tfrac{9}{2}x \\ &= -\tfrac{1}{2}x(4x^2 - 12x + 9) \\ &= -\tfrac{1}{2}x(2x - 3)^2 \end{aligned}$$

you can see that the zeros of $f$ are $x = 0$ (odd multiplicity) and $x = \frac{3}{2}$ (even multiplicity). So, the $x$-intercepts occur at $(0, 0)$ and $\left(\frac{3}{2}, 0\right)$. Add these points to your graph, as shown in Figure 2.22.

3. *Plot a Few Additional Points.* Use the zeros of the polynomial to find the test intervals. In each test interval, choose a representative $x$-value and evaluate the polynomial function, as shown in the table.

| Test interval | Representative $x$-value | Value of $f$ | Sign | Point on graph |
|---|---|---|---|---|
| $(-\infty, 0)$ | $-0.5$ | $f(-0.5) = 4$ | Positive | $(-0.5, 4)$ |
| $\left(0, \frac{3}{2}\right)$ | 0.5 | $f(0.5) = -1$ | Negative | $(0.5, -1)$ |
| $\left(\frac{3}{2}, \infty\right)$ | 2 | $f(2) = -1$ | Negative | $(2, -1)$ |

4. *Draw the Graph.* Draw a continuous curve through the points, as shown in Figure 2.23. As indicated by the multiplicities of the zeros, the graph crosses the $x$-axis at $(0, 0)$ but does not cross the $x$-axis at $\left(\frac{3}{2}, 0\right)$.

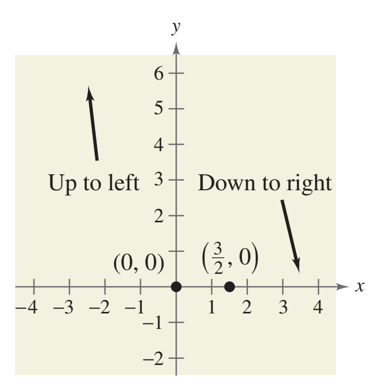



FIGURE 2.22

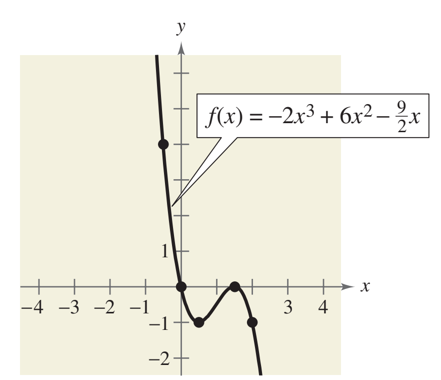



FIGURE 2.23

CHECKPOINT Now try Exercise 69.

**STUDY TIP**

Observe in Example 5 that the sign of $f(x)$ is positive to the left of and negative to the right of the zero $x = 0$. Similarly, the sign of $f(x)$ is negative to the left and to the right of the zero $x = \frac{3}{2}$. This suggests that if the zero of a polynomial function is of *odd* multiplicity, then the sign of $f(x)$ changes from one side of the zero to the other side. If the zero is of *even* multiplicity, then the sign of $f(x)$ does not change from one side of the zero to the other side.

**Activities**

1. Find all of the real zeros of $f(x) = 6x^4 - 33x^3 - 18x^2$.
   *Answer:* $-\frac{1}{2}, 0, 6$
2. Determine the right-hand and left-hand behavior of $f(x) = 6x^4 - 33x^3 - 18x^2$.
   *Answer:* The graph rises to the left and to the right.
3. Find a polynomial function of degree 3 that has zeros of 0, 2, and $-\frac{1}{3}$.
   *Answer:* $f(x) = 3x^3 - 5x^2 - 2x$

## The Intermediate Value Theorem

The next theorem, called the **Intermediate Value Theorem,** illustrates the existence of real zeros of polynomial functions. This theorem implies that if $(a, f(a))$ and $(b, f(b))$ are two points on the graph of a polynomial function such that $f(a) \neq f(b)$, then for any number $d$ between $f(a)$ and $f(b)$ there must be a number $c$ between $a$ and $b$ such that $f(c) = d$. (See Figure 2.24.)

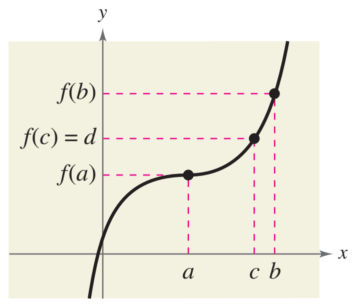



FIGURE 2.24

### Intermediate Value Theorem

Let $a$ and $b$ be real numbers such that $a < b$. If $f$ is a polynomial function such that $f(a) \neq f(b)$, then, in the interval $[a, b]$, $f$ takes on every value between $f(a)$ and $f(b)$.

The Intermediate Value Theorem helps you locate the real zeros of a polynomial function in the following way. If you can find a value $x = a$ at which a polynomial function is positive, and another value $x = b$ at which it is negative, you can conclude that the function has at least one real zero between these two values. For example, the function given by $f(x) = x^3 + x^2 + 1$ is negative when $x = -2$ and positive when $x = -1$. Therefore, it follows from the Intermediate Value Theorem that $f$ must have a real zero somewhere between $-2$ and $-1$, as shown in Figure 2.25.

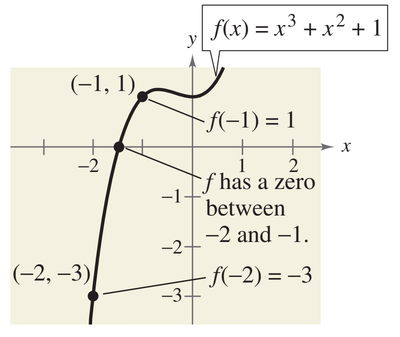



FIGURE 2.25

By continuing this line of reasoning, you can approximate any real zeros of a polynomial function to any desired accuracy. This concept is further demonstrated in Example 6.

The approximation process in Example 6 adapts very well to a graphing utility. Have students repeatedly use the *zoom* and *trace* features to find that the real zero of $f(x) = x^3 - x^2 + 1$ occurs between $-0.755$ and $-0.754$.

## Example 6 Approximating a Zero of a Polynomial Function

Use the Intermediate Value Theorem to approximate the real zero of

$$f(x) = x^3 - x^2 + 1.$$

### Solution

Begin by computing a few function values, as follows.

| $x$ | $f(x)$ |
|---|---|
| $-2$ | $-11$ |
| $-1$ | $-1$ |
| $0$ | $1$ |
| $1$ | $1$ |

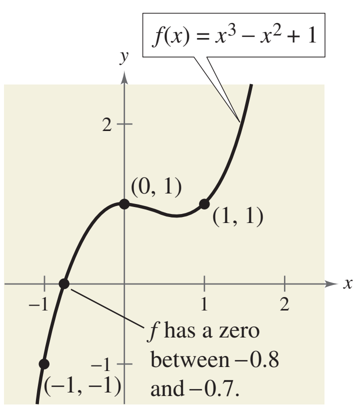


FIGURE 2.26

Because $f(-1)$ is negative and $f(0)$ is positive, you can apply the Intermediate Value Theorem to conclude that the function has a zero between $-1$ and 0. To pinpoint this zero more closely, divide the interval $[-1, 0]$ into tenths and evaluate the function at each point. When you do this, you will find that

$$f(-0.8) = -0.152 \quad \text{and} \quad f(-0.7) = 0.167.$$

So, $f$ must have a zero between $-0.8$ and $-0.7$, as shown in Figure 2.26. For a more accurate approximation, compute function values between $f(-0.8)$ and $f(-0.7)$ and apply the Intermediate Value Theorem again. By continuing this process, you can approximate this zero to any desired accuracy.

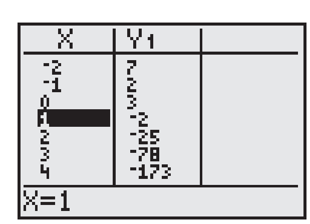
CHECKPOINT Now try Exercise 85.

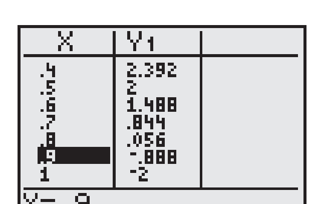
## Technology

You can use the *table* feature of a graphing utility to approximate the zeros of a polynomial function. For instance, for the function given by

$$f(x) = -2x^3 - 3x^2 + 3$$

create a table that shows the function values for $-20 \le x \le 20$, as shown in the first table at the right. Scroll through the table looking for consecutive function values that differ in sign. From the table, you can see that $f(0)$ and $f(1)$ differ in sign. So, you can conclude from the Intermediate Value Theorem that the function has a zero between 0 and 1. You can adjust your table to show function values for $0 \le x \le 1$ using increments of 0.1, as shown in the second table at the right. By scrolling through the table you can see that $f(0.8)$ and $f(0.9)$ differ in sign. So, the function has a zero between 0.8 and 0.9. If you repeat this process several times, you should obtain $x \approx 0.806$ as the zero of the function. Use the *zero* or *root* feature of a graphing utility to confirm this result.



## 2.2 Exercises

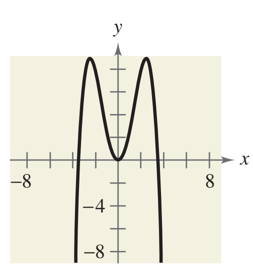
**VOCABULARY CHECK:** Fill in the blanks.

1. The graphs of all polynomial functions are ________, which means that the graphs have no breaks, holes, or gaps.
2. The ________ ________ ________ is used to determine the left-hand and right-hand behavior of the graph of a polynomial function.
3. A polynomial function of degree $n$ has at most ________ real zeros and at most ________ turning points.
4. If $x = a$ is a zero of a polynomial function $f$, then the following three statements are true.
   (a) $x = a$ is a ________ of the polynomial equation $f(x) = 0$.
   (b) ________ is a factor of the polynomial $f(x)$.
   (c) $(a, 0)$ is an ________ of the graph $f$.
5. If a real zero of a polynomial function is of even multiplicity, then the graph of $f$ ________ the $x$-axis at $x = a$, and if it is of odd multiplicity then the graph of $f$ ________ the $x$-axis at $x = a$.
6. A polynomial function is written in ________ form if its terms are written in descending order of exponents from left to right.
7. The ________ ________ Theorem states that if $f$ is a polynomial function such that $f(a) \neq f(b)$, then in the interval $[a, b]$, $f$ takes on every value between $f(a)$ and $f(b)$.

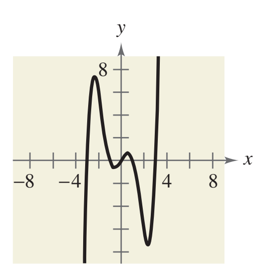
**PREREQUISITE SKILLS REVIEW:** Practice and review algebra skills needed for this section at **www.Eduspace.com.**

**In Exercises 1–8, match the polynomial function with its graph. [The graphs are labeled (a), (b), (c), (d), (e), (f), (g), and (h).]**

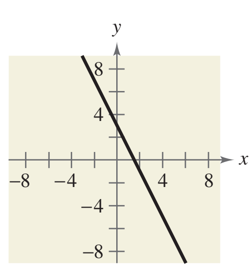
(a)



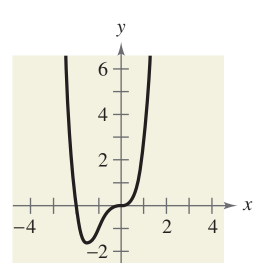

(b)



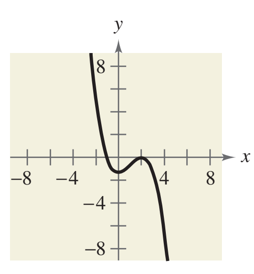

(c)



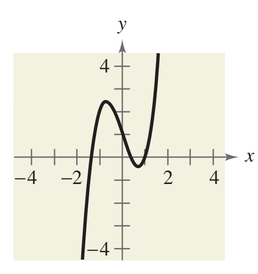

(d)



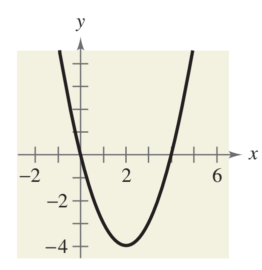

(e)



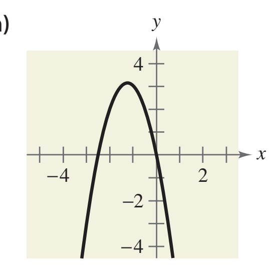

(f)




(g)




(h)




1. $f(x) = -2x + 3$
2. $f(x) = x^2 - 4x$
3. $f(x) = -2x^2 - 5x$
4. $f(x) = 2x^3 - 3x + 1$
5. $f(x) = -\frac{1}{4}x^4 + 3x^2$
6. $f(x) = -\frac{1}{3}x^3 + x^2 - \frac{4}{3}$
7. $f(x) = x^4 + 2x^3$
8. $f(x) = \frac{1}{5}x^5 - 2x^3 + \frac{9}{5}x$

**In Exercises 9–12, sketch the graph of $y = x^n$ and each transformation.**

9. $y = x^3$
   (a) $f(x) = (x - 2)^3$
   (b) $f(x) = x^3 - 2$
   (c) $f(x) = -\frac{1}{2}x^3$
   (d) $f(x) = (x - 2)^3 - 2$
10. $y = x^5$
   (a) $f(x) = (x + 1)^5$
   (b) $f(x) = x^5 + 1$
   (c) $f(x) = 1 - \frac{1}{2}x^5$
   (d) $f(x) = -\frac{1}{2}(x + 1)^5$
11. $y = x^4$
   (a) $f(x) = (x + 3)^4$
   (b) $f(x) = x^4 - 3$
   (c) $f(x) = 4 - x^4$
   (d) $f(x) = \frac{1}{2}(x - 1)^4$
   (e) $f(x) = (2x)^4 + 1$
   (f) $f(x) = \left(\frac{1}{2}x\right)^4 - 2$

**12.** $y = x^6$

(a) $f(x) = -\frac{1}{8}x^6$ (b) $f(x) = (x + 2)^6 - 4$

(c) $f(x) = x^6 - 4$ (d) $f(x) = -\frac{1}{4}x^6 + 1$

(e) $f(x) = \left(\frac{1}{4}x\right)^6 - 2$ (f) $f(x) = (2x)^6 - 1$

**In Exercises 13–22, describe the right-hand and left-hand behavior of the graph of the polynomial function.**

**13.** $f(x) = \frac{1}{3}x^3 + 5x$ **14.** $f(x) = 2x^2 - 3x + 1$

**15.** $g(x) = 5 - \frac{7}{2}x - 3x^2$ **16.** $h(x) = 1 - x^6$

**17.** $f(x) = -2.1x^5 + 4x^3 - 2$

**18.** $f(x) = 2x^5 - 5x + 7.5$

**19.** $f(x) = 6 - 2x + 4x^2 - 5x^3$

**20.** $f(x) = \dfrac{3x^4 - 2x + 5}{4}$

**21.** $h(t) = -\frac{2}{3}(t^2 - 5t + 3)$

**22.** $f(s) = -\frac{7}{8}(s^3 + 5s^2 - 7s + 1)$

***Graphical Analysis*** **In Exercises 23–26, use a graphing utility to graph the functions *f* and *g* in the same viewing window. Zoom out sufficiently far to show that the right-hand and left-hand behaviors of *f* and *g* appear identical.**

**23.** $f(x) = 3x^3 - 9x + 1, \quad g(x) = 3x^3$

**24.** $f(x) = -\frac{1}{3}(x^3 - 3x + 2), \quad g(x) = -\frac{1}{3}x^3$

**25.** $f(x) = -(x^4 - 4x^3 + 16x), \quad g(x) = -x^4$

**26.** $f(x) = 3x^4 - 6x^2, \quad g(x) = 3x^4$

**In Exercises 27–42, (a) find all the real zeros of the polynomial function, (b) determine the multiplicity of each zero and the number of turning points of the graph of the function, and (c) use a graphing utility to graph the function and verify your answers.**

**27.** $f(x) = x^2 - 25$ **28.** $f(x) = 49 - x^2$

**29.** $h(t) = t^2 - 6t + 9$ **30.** $f(x) = x^2 + 10x + 25$

**31.** $f(x) = \frac{1}{3}x^2 + \frac{1}{3}x - \frac{2}{3}$

**32.** $f(x) = \frac{1}{2}x^2 + \frac{5}{2}x - \frac{3}{2}$

**33.** $f(x) = 3x^3 - 12x^2 + 3x$

**34.** $g(x) = 5x(x^2 - 2x - 1)$

**35.** $f(t) = t^3 - 4t^2 + 4t$

**36.** $f(x) = x^4 - x^3 - 20x^2$

**37.** $g(t) = t^5 - 6t^3 + 9t$

**38.** $f(x) = x^5 + x^3 - 6x$

**39.** $f(x) = 5x^4 + 15x^2 + 10$

**40.** $f(x) = 2x^4 - 2x^2 - 40$

**41.** $g(x) = x^3 + 3x^2 - 4x - 12$

**42.** $f(x) = x^3 - 4x^2 - 25x + 100$

***Graphical Analysis*** **In Exercises 43–46, (a) use a graphing utility to graph the function, (b) use the graph to approximate any *x*-intercepts of the graph, (c) set $y = 0$ and solve the resulting equation, and (d) compare the results of part (c) with any *x*-intercepts of the graph.**

**43.** $y = 4x^3 - 20x^2 + 25x$

**44.** $y = 4x^3 + 4x^2 - 8x + 8$

**45.** $y = x^5 - 5x^3 + 4x$

**46.** $y = \frac{1}{4}x^3(x^2 - 9)$

**In Exercises 47–56, find a polynomial function that has the given zeros. (There are many correct answers.)**

**47.** 0, 10 **48.** 0, $-3$

**49.** 2, $-6$ **50.** $-4$, 5

**51.** 0, $-2$, $-3$ **52.** 0, 2, 5

**53.** 4, $-3$, 3, 0 **54.** $-2$, $-1$, 0, 1, 2

**55.** $1 + \sqrt{3}$, $1 - \sqrt{3}$ **56.** 2, $4 + \sqrt{5}$, $4 - \sqrt{5}$

**In Exercises 57–66, find a polynomial of degree *n* that has the given zero(s). (There are many correct answers.)**

| | *Zero(s)* | *Degree* |
|---|---|---|
| **57.** | $x = -2$ | $n = 2$ |
| **58.** | $x = -8, -4$ | $n = 2$ |
| **59.** | $x = -3, 0, 1$ | $n = 3$ |
| **60.** | $x = -2, 4, 7$ | $n = 3$ |
| **61.** | $x = 0, \sqrt{3}, -\sqrt{3}$ | $n = 3$ |
| **62.** | $x = 9$ | $n = 3$ |
| **63.** | $x = -5, 1, 2$ | $n = 4$ |
| **64.** | $x = -4, -1, 3, 6$ | $n = 4$ |
| **65.** | $x = 0, -4$ | $n = 5$ |
| **66.** | $x = -3, 1, 5, 6$ | $n = 5$ |

**In Exercises 67–80, sketch the graph of the function by (a) applying the Leading Coefficient Test, (b) finding the zeros of the polynomial, (c) plotting sufficient solution points, and (d) drawing a continuous curve through the points.**

**67.** $f(x) = x^3 - 9x$ **68.** $g(x) = x^4 - 4x^2$

**69.** $f(t) = \frac{1}{4}(t^2 - 2t + 15)$

**70.** $g(x) = -x^2 + 10x - 16$

**71.** $f(x) = x^3 - 3x^2$ **72.** $f(x) = 1 - x^3$

**73.** $f(x) = 3x^3 - 15x^2 + 18x$

**74.** $f(x) = -4x^3 + 4x^2 + 15x$

**75.** $f(x) = -5x^2 - x^3$ **76.** $f(x) = -48x^2 + 3x^4$

**77.** $f(x) = x^2(x - 4)$ **78.** $h(x) = \frac{1}{3}x^3(x - 4)^2$

**79.** $g(t) = -\frac{1}{4}(t - 2)^2(t + 2)^2$

**80.** $g(x) = \frac{1}{10}(x + 1)^2(x - 3)^3$

**In Exercises 81–84, use a graphing utility to graph the function. Use the *zero* or *root* feature to approximate the real zeros of the function. Then determine the multiplicity of each zero.**

**81.** $f(x) = x^3 - 4x$

**82.** $f(x) = \frac{1}{4}x^4 - 2x^2$

**83.** $g(x) = \frac{1}{5}(x + 1)^2(x - 3)(2x - 9)$

**84.** $h(x) = \frac{1}{5}(x + 2)^2(3x - 5)^2$

**In Exercises 85–88, use the Intermediate Value Theorem and the *table* feature of a graphing utility to find intervals one unit in length in which the polynomial function is guaranteed to have a zero. Adjust the table to approximate the zeros of the function. Use the *zero* or *root* feature of a graphing utility to verify your results.**

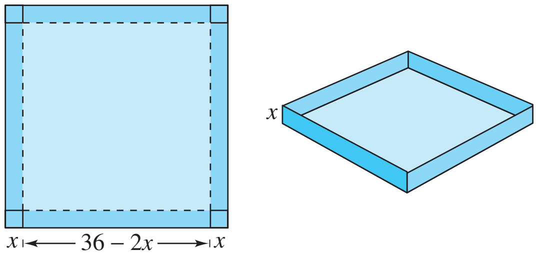
**85.** $f(x) = x^3 - 3x^2 + 3$

**86.** $f(x) = 0.11x^3 - 2.07x^2 + 9.81x - 6.88$

**87.** $g(x) = 3x^4 + 4x^3 - 3$

**88.** $h(x) = x^4 - 10x^2 + 3$

**89.** ***Numerical and Graphical Analysis*** An open box is to be made from a square piece of material, 36 inches on a side, by cutting equal squares with sides of length $x$ from the corners and turning up the sides (see figure).




(a) Verify that the volume of the box is given by the function

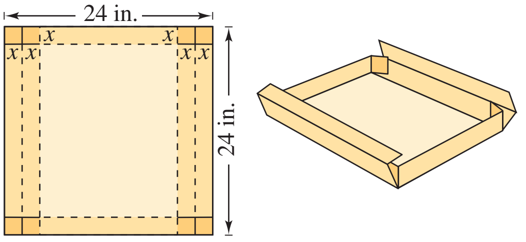
$$V(x) = x(36 - 2x)^2.$$

(b) Determine the domain of the function.

(c) Use a graphing utility to create a table that shows the box height $x$ and the corresponding volumes $V$. Use the table to estimate the dimensions that will produce a maximum volume.

(d) Use a graphing utility to graph $V$ and use the graph to estimate the value of $x$ for which $V(x)$ is maximum. Compare your result with that of part (c).

**90.** ***Maximum Volume*** An open box with locking tabs is to be made from a square piece of material 24 inches on a side. This is to be done by cutting equal squares from the corners and folding along the dashed lines shown in the figure.



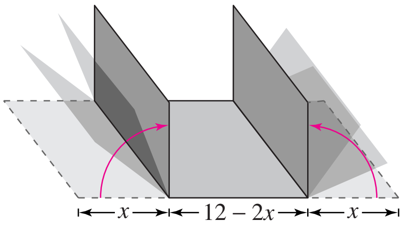

(a) Verify that the volume of the box is given by the function

$$V(x) = 8x(6 - x)(12 - x).$$

(b) Determine the domain of the function $V$.

(c) Sketch a graph of the function and estimate the value of $x$ for which $V(x)$ is maximum.

**91.** ***Construction*** A roofing contractor is fabricating gutters from 12-inch aluminum sheeting. The contractor plans to use an aluminum siding folding press to create the gutter by creasing equal lengths for the sidewalls (see figure).




(a) Let $x$ represent the height of the sidewall of the gutter. Write a function $A$ that represents the cross-sectional area of the gutter.

(b) The length of the aluminum sheeting is 16 feet. Write a function $V$ that represents the volume of one run of gutter in terms of $x$.

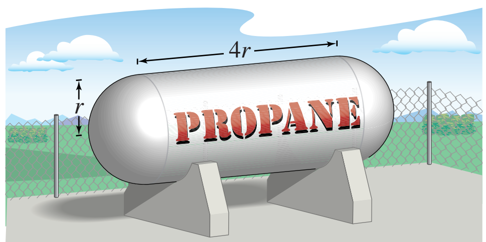
(c) Determine the domain of the function in part (b).

(d) Use a graphing utility to create a table that shows the sidewall height $x$ and the corresponding volumes $V$. Use the table to estimate the dimensions that will produce a maximum volume.

(e) Use a graphing utility to graph $V$. Use the graph to estimate the value of $x$ for which $V(x)$ is a maximum. Compare your result with that of part (d).

(f) Would the value of $x$ change if the aluminum sheeting were of different lengths? Explain.

**92.** ***Construction*** An industrial propane tank is formed by adjoining two hemispheres to the ends of a right circular cylinder. The length of the cylindrical portion of the tank is four times the radius of the hemispherical components (see figure).




(a) Write a function that represents the total volume $V$ of the tank in terms of $r$.

(b) Find the domain of the function.

(c) Use a graphing utility to graph the function.

(d) The total volume of the tank is to be 120 cubic feet. Use the graph from part (c) to estimate the radius and length of the cylindrical portion of the tank.

***Data Analysis: Home Prices*** **In Exercise 93–96, use the table, which shows the median prices (in thousands of dollars) of new privately owned U.S. homes in the Midwest $y_1$ and in the South $y_2$ for the years 1997 through 2003. The data can be approximated by the following models.**

$$y_1 = 0.139t^3 - 4.42t^2 + 51.1t - 39$$

$$y_2 = 0.056t^3 - 1.73t^2 + 23.8t + 29$$

**In the models, $t$ represents the year, with $t = 7$ corresponding to 1997.** (Source: U.S. Census Bureau; U.S. Department of Housing and Urban Development)

| Year, $t$ | $y_1$ | $y_2$ |
|---|---|---|
| 7 | 150 | 130 |
| 8 | 158 | 136 |
| 9 | 164 | 146 |
| 10 | 170 | 148 |
| 11 | 173 | 155 |
| 12 | 178 | 163 |
| 13 | 184 | 168 |

**93.** Use a graphing utility to plot the data and graph the model for $y_1$ in the same viewing window. How closely does the model represent the data?

**94.** Use a graphing utility to plot the data and graph the model for $y_2$ in the same viewing window. How closely does the model represent the data?

**95.** Use the models to predict the median prices of a new privately owned home in both regions in 2008. Do your answers seem reasonable? Explain.

**96.** Use the graphs of the models in Exercises 93 and 94 to write a short paragraph about the relationship between the median prices of homes in the two regions.

## Model It

**97.** ***Tree Growth*** The growth of a red oak tree is approximated by the function

$$G = -0.003t^3 + 0.137t^2 + 0.458t - 0.839$$

where $G$ is the height of the tree (in feet) and $t$ $(2 \le t \le 34)$ is its age (in years).

(a) Use a graphing utility to graph the function. (*Hint:* Use a viewing window in which $-10 \le x \le 45$ and $-5 \le y \le 60$.)

(b) Estimate the age of the tree when it is growing most rapidly. This point is called the *point of diminishing returns* because the increase in size will be less with each additional year.

(c) Using calculus, the point of diminishing returns can also be found by finding the vertex of the parabola given by

$$y = -0.009t^2 + 0.274t + 0.458.$$

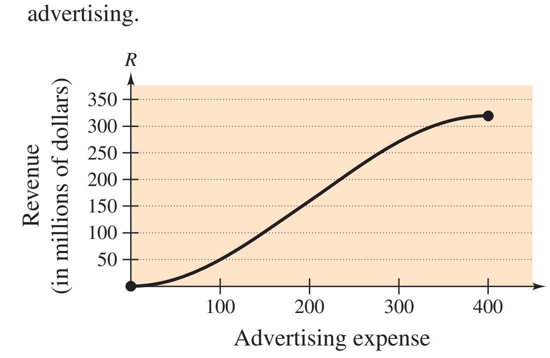
Find the vertex of this parabola.

(d) Compare your results from parts (b) and (c).

**98.** ***Revenue*** The total revenue $R$ (in millions of dollars) for a company is related to its advertising expense by the function

$$R = \frac{1}{100{,}000}(-x^3 + 600x^2), \qquad 0 \le x \le 400$$

where $x$ is the amount spent on advertising (in tens of thousands of dollars). Use the graph of this function, shown in the figure, to estimate the point on the graph at which the function is increasing most rapidly. This point is called the *point of diminishing returns* because any expense above this amount will yield less return per dollar invested in advertising.



## Synthesis

***True or False?*** **In Exercises 99–101, determine whether the statement is true or false. Justify your answer.**

**99.** A fifth-degree polynomial can have five turning points in its graph.

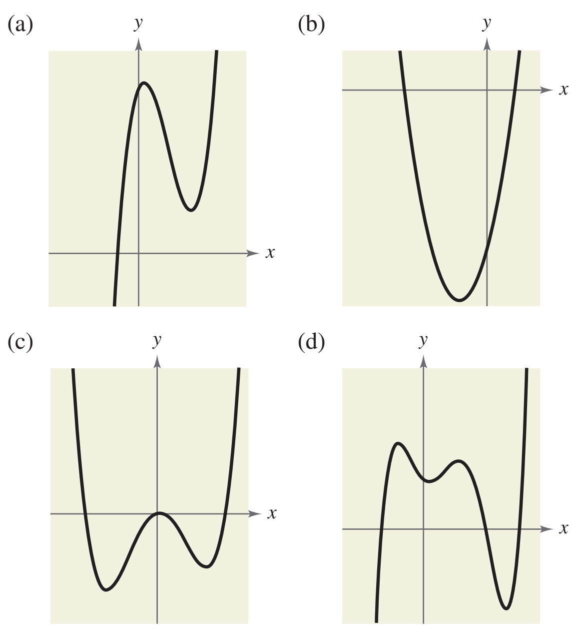
**100.** It is possible for a sixth-degree polynomial to have only one solution.

**101.** The graph of the function given by

$$f(x) = 2 + x - x^2 + x^3 - x^4 + x^5 + x^6 - x^7$$

rises to the left and falls to the right.

**102.** ***Graphical Analysis*** For each graph, describe a polynomial function that could represent the graph. (Indicate the degree of the function and the sign of its leading coefficient.)




**103.** ***Graphical Reasoning*** Sketch a graph of the function given by $f(x) = x^4$. Explain how the graph of each function $g$ differs (if it does) from the graph of each function $f$. Determine whether $g$ is odd, even, or neither.

(a) $g(x) = f(x) + 2$

(b) $g(x) = f(x + 2)$

(c) $g(x) = f(-x)$

(d) $g(x) = -f(x)$

(e) $g(x) = f\left(\frac{1}{2}x\right)$

(f) $g(x) = \frac{1}{2}f(x)$

(g) $g(x) = f\left(x^{3/4}\right)$

(h) $g(x) = (f \circ f)(x)$

**104.** ***Exploration*** Explore the transformations of the form $g(x) = a(x - h)^5 + k$.

(a) Use a graphing utility to graph the functions given by

$$y_1 = -\frac{1}{3}(x - 2)^5 + 1$$

and

$$y_2 = \frac{3}{5}(x + 2)^5 - 3.$$

Determine whether the graphs are increasing or decreasing. Explain.

(b) Will the graph of $g$ always be increasing or decreasing? If so, is this behavior determined by $a$, $h$, or $k$? Explain.

(c) Use a graphing utility to graph the function given by

$$H(x) = x^5 - 3x^3 + 2x + 1.$$

Use the graph and the result of part (b) to determine whether $H$ can be written in the form $H(x) = a(x - h)^5 + k$. Explain.

## Skills Review

**In Exercises 105–108, factor the expression completely.**

**105.** $5x^2 + 7x - 24$

**106.** $6x^3 - 61x^2 + 10x$

**107.** $4x^4 - 7x^3 - 15x^2$

**108.** $y^3 + 216$

**In Exercises 109–112, solve the equation by factoring.**

**109.** $2x^2 - x - 28 = 0$

**110.** $3x^2 - 22x - 16 = 0$

**111.** $12x^2 + 11x - 5 = 0$

**112.** $x^2 + 24x + 144 = 0$

**In Exercises 113–116, solve the equation by completing the square.**

**113.** $x^2 - 2x - 21 = 0$

**114.** $x^2 - 8x + 2 = 0$

**115.** $2x^2 + 5x - 20 = 0$

**116.** $3x^2 + 4x - 9 = 0$

**In Exercises 117–122, describe the transformation from a common function that occurs in $f(x)$. Then sketch its graph.**

**117.** $f(x) = (x + 4)^2$

**118.** $f(x) = 3 - x^2$

**119.** $f(x) = \sqrt{x + 1} - 5$

**120.** $f(x) = 7 - \sqrt{x - 6}$

**121.** $f(x) = 2[\![x]\!] + 9$

**122.** $f(x) = 10 - \frac{1}{3}[\![x + 3]\!]$