Unit 6.3

Evaluating Definite Integrals
Properties of Definite Integrals

- How do we convert Riemann Sum Notation to Integral Notation?
- What are two methods to evaluate a definite integral without approximations?
- What are the four properties of definite integrals?
- How do properties of integrals allow us to evaluate definite integrals?

HOW CAN RECTANGULAR APPROXIMATIONS MEHTODS BECOME MORE ACCURATE?

Area = $\Delta x f(x_i)$

Conversion Rule:

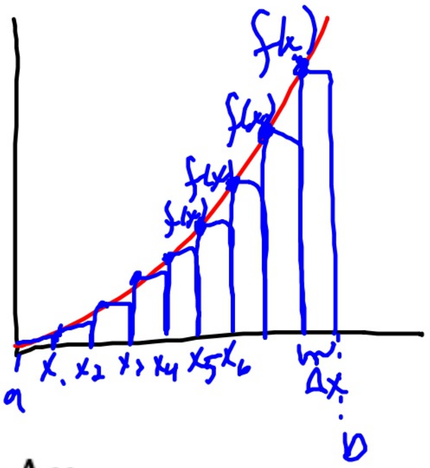
Infinitely many
Infinitely thin

upper bound

$$\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\,\Delta x = \int_a^b f(x)\,dx$$

lower bound

Integral of f(x) from a to b



NOTE: $\Delta x = \dfrac{b-a}{n}$ $\qquad x_i = a + i \cdot \Delta x$

$\Delta x = \dfrac{b-a}{n} \approx 0$

EXAMPLE 1: **Converting from**

Integral Notation ⇒ Riemann Sum Notation

$$\int_{0}^{3} x^3 dx$$

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\, \Delta x$$

$f(x) = x^3$

$\Delta x = \frac{b-a}{n} = \frac{3-0}{n} = \frac{3}{n}$

$x_i = a + i\Delta x = 0 + i\left(\frac{3}{n}\right) = \frac{3i}{n}$

$$\lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{3i}{n}\right)^3 \cdot \left(\frac{3}{n}\right)$$

EXAMPLE 2: **Converting from Integral Notation ⇨ Riemann Sum Notation**

$$\int_{-2_a}^{3^b}(1-5x^2)\,dx$$

$$f(x)=1-5x^2$$

$$\Delta x=\frac{b-a}{n}=\frac{3-(-2)}{n}=\frac{5}{n}$$

$$x_i=a+i\Delta x=-2+i\left(\frac{5}{n}\right)=\frac{5i}{n}-2$$

$$\lim_{n\to\infty}\sum_{i=1}^{n}f(x_i)\Delta x$$

$$\lim_{n\to\infty}\sum_{i=1}^{n}\left(1-5\left(\frac{5i}{n}-2\right)^2\right)\cdot\frac{5}{n}$$

EXAMPLE 3: **Converting from Integral Notation ⇒ Riemann Sum Notation**

$$\int_{\pi}^{2\pi} \cos(x)\, dx$$

b, a

$f(x) = \cos(x)$

$\Delta x = \frac{b-a}{n} = \frac{2\pi - \pi}{n} = \frac{\pi}{n}$

$x_i = a + i\Delta x = \pi + \frac{\pi i}{n}$

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

$$\lim_{n\to\infty} \sum_{i=1}^{n} \cos\left(\pi + \frac{\pi i}{n}\right) \cdot \frac{\pi}{n}$$

EXAMPLE 4: **Converting from**

Riemann Sum Notation ⇨ Integral Notation

$$\lim_{n\to\infty} \sum_{i=1}^{n} \underbrace{\left(4\left(\frac{5i}{n}\right)\right)}_{f(x_i)} \cdot \underbrace{\frac{5}{n}}_{\Delta x} = \int_a^b f(x)\,dx = \int_0^5 4x\,dx$$

$x_i = a + i\Delta x$

$\frac{5i}{n} = a + i\left(\frac{5}{n}\right)$

$\frac{5i}{n} = a + \frac{5i}{n}$

$0 = a$

$\Delta x = \frac{b-a}{n}$

$\frac{5}{n} = \frac{b-a}{n}$

$\frac{5}{n} = \frac{b-0}{n}$

$\frac{5}{n} = \frac{b}{n}$

$5 = b$

EXAMPLE 5: Converting from Riemann Sum Notation ⇨ Integral Notation

$$\lim_{n\to\infty} \sum_{i=1}^{n} \underbrace{\left(2\left(4+\frac{2i}{n}\right)^2\right)}_{f(x_i)} \cdot \underbrace{\frac{2}{n}}_{\Delta x} = \int_a^b f(x)\,dx = \int_4^6 2x^2\,dx$$

$x_i = a + i\Delta x$

$4 + \frac{2i}{n} = a + i\left(\frac{2}{n}\right)$

$4 + \frac{2i}{n} = a + \frac{2i}{n}$

$4 = a$

$\Delta x = \frac{b-a}{n}$

$\frac{2}{n} = \frac{b-a}{n}$

$\frac{2}{n} = \frac{b-4}{n}$

$2 = b-4$

$6 = b$

EXAMPLE 6: Converting from Riemann Sum Notation ⇨ Integral Notation

$$\lim_{n\to\infty} \sum_{i=1}^{n} \underbrace{\left(\left(\frac{4i}{n}-5\right)^2+3\left(\frac{4i}{n}-5\right)+5\right)}_{f(x_i)} \cdot \underbrace{\frac{4}{n}}_{\Delta x} = \int_a^b f(x)\,dx = \int_{-5}^{-1} x^2+3x+5\,dx$$

$x_i = a + i\Delta x$

$\frac{4i}{n} - 5 = a + i\left(\frac{4}{n}\right)$

$\frac{4i}{n} - 5 = a + \frac{4i}{n}$

$-5 = a$

$\Delta x = \frac{b-a}{n}$

$\frac{4}{n} = \frac{b-a}{n}$

$\frac{4}{n} = \frac{b-(-5)}{n}$

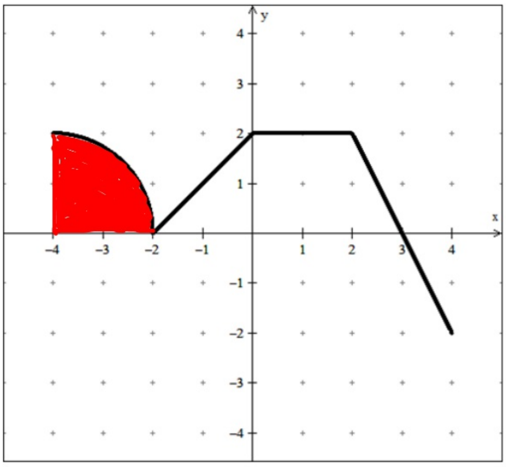
$\frac{4}{n} = \frac{b+5}{n}$

$4 = b+5$

$-1 = b$

Determining the exact value of the integral without using Approximations

METHOD#1: Find the EXACT Area

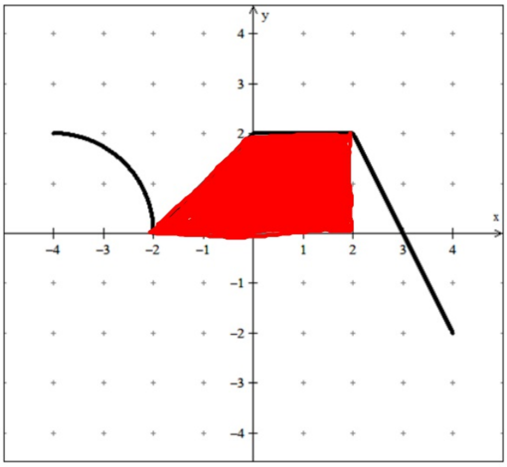


1.) $\int_{-4}^{-2} f(x)dx = \frac{\pi r^2}{4} = \frac{\pi(2)^2}{4}$

$= \pi$

Determining the exact value of the integral without using Approximations

METHOD#1: Find the EXACT Area



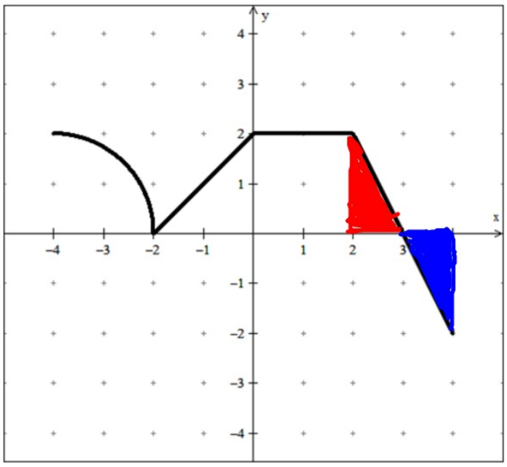
2.) $\int_{-2}^{2} f(x)dx = \frac{1}{2}(b_1+b_2)h$

$= \frac{1}{2}(4+2)(2)$

$= 6$

Determining the exact value of the integral without using Approximations

METHOD#1: Find the EXACT Area



3.) $\int_2^4 f(x)dx$

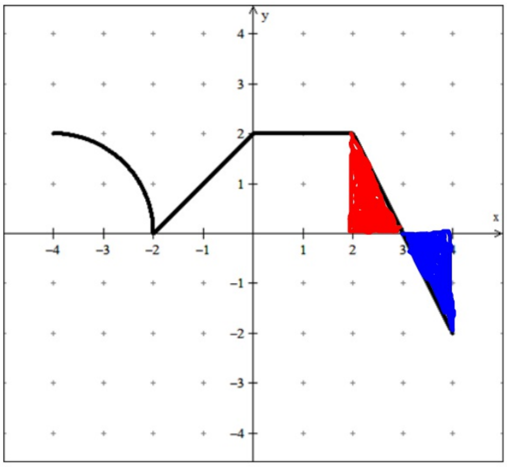
$$= \int_2^3 f(x)dx + \int_3^4 f(x)dx$$

$$= \frac{1}{2}(1)(2) + -\left(\frac{1}{2}(1)(2)\right)$$

$$= 0$$

Determining the exact value of the integral without using Approximations

METHOD#1: Find the EXACT Area



4.) $\int_2^4 |f(x)|dx$

$= \int_2^3 |f(x)|dx + \int_3^4 |f(x)|dx$

$= \left|\frac{1}{2}(1)(2)\right| + \left|-\left(\frac{1}{2}(1)(2)\right)\right|$

$= 1 + 1 = 2$

Determining the exact value of the integral without using Approximations

METHOD#2: Using Properties of Integrals

ADDITION PROPERTY If a < b < c, then...

$$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$$

COEFFICIENT PROPERTY

For any Real Number, c,

$$\boxed{\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx}$$

BOUNDS PROPERTY

Bounds should always be least ~~to~~ greatest!

$$\boxed{\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx}$$

INTEGRAL SUM/DIFFERENCE PROPERTY

$$\boxed{\int_a^b \left(f(x) \pm g(x)\right) dx = \int_a^b f(x)\, dx \pm \int_a^b g(x)\, dx}$$

EXAMPLE 1: **Use the given and definite integral properties to solve**

GIVEN: $\int_0^2 f(x)dx = 5$, $\int_2^4 f(x)dx = 3$, $\int_2^6 f(x)dx = 12$

a.) $\int_0^6 f(x)dx =$

$\int_0^2 f(x)dx + \int_2^6 f(x)dx$

$= 5 + 12 = 17$

b.) $\int_6^2 f(x)dx = -\int_2^6 f(x)dx$

Has to be least to greatest!

$= -12$

EXAMPLE 1: **Use the given and definite integral properties to solve**

GIVEN: $\int_0^2 f(x)dx=5$, $\int_2^4 f(x)dx = 3$, $\int_2^6 f(x)dx=12$

c.) $\int_0^2 4f(x)dx =$

$4\int_0^2 f(x)dx = 4(5) = 20$

d.) $\int_6^0 2f(x)dx =$

$2\int_6^0 f(x)dx = -2\int_0^6 f(x)dx$

$-2\left[\int_0^2 f(x)dx + \int_2^6 f(x)dx\right]$

$-2[5+12] = -2(17) = -34$

EXAMPLE 1: Use the given and definite integral properties to solve

GIVEN: $\int_0^2 f(x)dx = 5,\ \int_2^4 f(x)dx = 3,\ \int_2^6 f(x)dx = 12$

e.) $\int_4^6 [f(x)+2]dx =$

$$= \int_4^6 f(x)dx + \int_4^6 2dx$$

$$= \left(\int_2^6 f(x)dx - \int_2^4 f(x)dx\right) + \int_4^6 2dx$$

$$= (12-3) + \int_4^6 2dx$$

$$= 12 - 3 + 4$$

$$= 13$$

f.) $\int_4^0 -f(x)dx = -\int_0^4 -f(x)dx$

$$= \int_0^4 f(x)dx$$

$$= \int_0^2 f(x)dx + \int_2^4 f(x)dx$$

$$= 5 + 3 = 8$$

EXAMPLE 2: Use the given and definite integral properties to solve

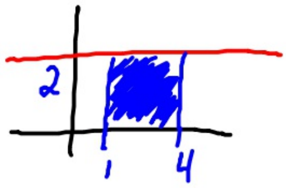
GIVEN: $\displaystyle\int_1^4 f(x)\,dx = 6$ and $\displaystyle\int_1^4 g(x)\,dx = 3$

a.) $\int_1^4 [3f(x) + 7g(x) + 2]dx$

$= \displaystyle\int_1^4 3f(x)dx + \int_1^4 7g(x)dx + \int_1^4 2dx$

$= 3\displaystyle\int_1^4 f(x)dx + 7\int_1^4 g(x)dx + \int_1^4 2dx$

$= 3(6) + 7(3) + \displaystyle\int_1^4 2dx$



$= 18 + 21 + 6$

$= 45$

EXAMPLE 2: Use the given and definite integral properties to solve

GIVEN: $\displaystyle\int_1^4 f(x)\,dx = 6$ and $\displaystyle\int_1^4 g(x)\,dx = 3$

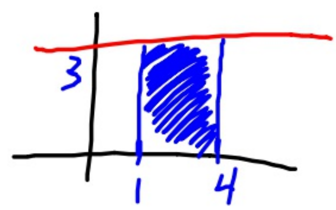
b.) $\displaystyle\int_4^1 [2g(x) - f(x) - 3]dx$

$$= -\int_1^4 (2g(x) - f(x) - 3)\,dx$$

$$= -\left[\int_1^4 2g(x)dx - \int_1^4 f(x)dx - \int_1^4 3dx\right]$$

$$= -\left[2\int_1^4 g(x)dx - \int_1^4 f(x)dx - \int_1^4 3dx\right]$$

$$= -\left[2(3) - (6) - \int_1^4 3dx\right]$$



$$= -[6 - 6 - 9]$$

$$= 9$$

SELF CHECK: Evaluating Definite Integrals
Properties Definite Integrals

- ☑ How do we convert Riemann Sum Notation to Integral Notation?
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