48. Use the intersect function on a graphing calculator to determine that the curves intersect at $x = \pm 1.8933$. A shell has radius x and height $3^{1-x^2} - \frac{x^2 - 3}{10}$. The volume, which is calculated using the *right half* of the area, is $\int_0^{1.8933} 2\pi(x)\left(3^{1-x^2} - \frac{x^2 - 3}{10}\right) dx$, which using NINT evaluates to ≈ 9.7717.

49. **(a)** $y = -\frac{5}{4}(x + 2)(x - 2) = 5 - \frac{5}{4}x^2$

(b) Revolve about the line $x = 4$, using cylindrical shells. A shell has radius $4 - x$ and height $5 - \frac{5}{4}x^2$. The total volume is

$$\int_{-2}^{2} 2\pi(4 - x)\left(5 - \frac{5}{4}x^2\right) dx$$
$$= 10\pi\int_{-2}^{2}\left(\frac{1}{4}x^3 - x^2 - x + 4\right) dx$$
$$= 10\pi\left[\frac{1}{16}x^4 - \frac{1}{3}x^3 - \frac{1}{2}x^2 + 4x\right]_{-2}^{2}$$
$$= \frac{320}{3}\pi \approx 335.1032 \text{ in}^3.$$

50. Since $\frac{dL}{dx} = \frac{1}{x} + f'(x)$ must equal $\sqrt{1 + (f'(x))^2}$,
$1 + (f'(x))^2 = \frac{1}{x^2} + \frac{2}{x}f'(x) + (f'(x))^2$, and
$f'(x) = \frac{1}{2}x - \frac{1}{2x}$. Then $f(x) = \frac{1}{4}x^2 - \frac{1}{2}\ln x + C$, and the requirement to pass through $(1, 1)$ means that $C = \frac{3}{4}$. The function is $f(x) = \frac{1}{4}x^2 - \frac{1}{2}\ln x + \frac{3}{4} = \frac{x^2 - 2\ln x + 3}{4}$.

51. $y' = \sec^2 x$, so the area is $\int_0^{\pi/4} 2\pi(\tan x)\sqrt{1 + (\sec^2 x)^2}\, dx$, which using NINT evaluates to ≈ 3.84.

52. $x = \frac{1}{y}$ and $x' = -\frac{1}{y^2}$, so the area is
$\int_1^2 2\pi\left(\frac{1}{y}\right)\sqrt{1 + \left(-\frac{1}{y^2}\right)^2}\, dy$,
which using NINT evaluates to ≈ 5.02.

Chapter 8
L'Hôpital's Rule, Improper Integrals, and Partial Fractions

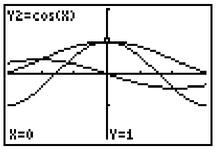
■ Section 8.1 L'Hôpital's Rule (pp. 417–425)

Exploration 1 Exploring L'Hôpital's Rule Graphically

1. $\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \frac{\cos x}{1} = 1$

2. The two graphs suggest that $\lim_{x\to 0} \frac{y_1}{y_2} = \lim_{x\to 0} \frac{y_1'}{y_2'}$.

3. $y_5 = \frac{x\cos x - \sin x}{x^2}$. The graphs of y_3 and y_5 clearly show that l'Hôpital's Rule does not say that $\lim_{x\to 0} \frac{y_1}{y_2}$ is equal to $\lim_{x\to 0} \left(\frac{y_1}{y_2}\right)'$.



[−3, 3] by [−2, 2]

Quick Review 8.1

1.

x	$\left(1 + \frac{0.1}{x}\right)^x$
1	1.1000
10	1.1046
100	1.1051
1000	1.1052
10,000	1.1052
1,000,000	1.1052

As $x\to\infty$, $\left(1 + \frac{0.1}{x}\right)^x$ approaches 1.1052.

2.

x	$x^{1/(\ln x)}$
0.1	2.7183
0.01	2.7183
0.001	2.7183
0.0001	2.7183
0.00001	2.7183

As $x\to 0^+$, $x^{1/(\ln x)}$ approaches 2.7183.

3.

x	$\left(1 - \frac{1}{x}\right)^x$
−1	0.5
−0.1	0.78679
−0.01	0.95490
−0.001	0.99312
−0.0001	0.99908
−0.00001	0.99988
−0.000001	0.99999

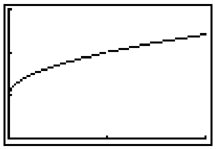
As $x\to 0^-$, $\left(1 - \frac{1}{x}\right)^x$ approaches 1.

4.

x	$\left(1 + \frac{1}{x}\right)^x$
−1.1	13.981
−1.01	105.77
−1.001	1007.9
−1.0001	10010

As $x\to -1^-$, $\left(1 + \frac{1}{x}\right)^x$ goes to ∞.

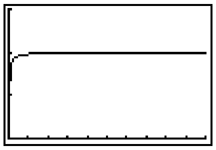
5.



[0, 2] by [0, 3]

As $t \to 1$, $\frac{t-1}{\sqrt{t}-1}$ approaches 2.

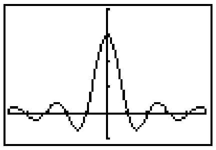
6.



[0, 500] by [0, 3]

As $x \to \infty$, $\frac{\sqrt{4x^2+1}}{x+1}$ approaches 2.

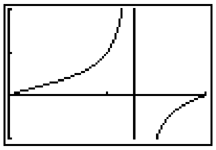
7.



[−5, 5] by [−1, 4]

As $x \to 0$, $\frac{\sin 3x}{x}$ approaches 3.

8.



[0, π] by [−1, 2]

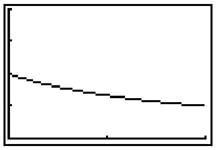
As $\theta \to \frac{\pi}{2}$, $\frac{\tan\theta}{2+\tan\theta}$ approaches 1.

9. $y = \frac{1}{h}\sin h$

10. $y = (1+h)^{1/h}$

Section 8.1 Exercises

1.

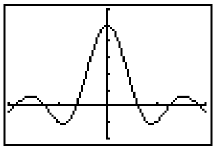


[0, 2] by [0, 1]

From the graph, the limit appears to be $\frac{1}{4}$.

$$\lim_{x\to2}\frac{x-2}{x^2-4} = \lim_{x\to2}\frac{1}{2x} = \frac{1}{4}$$

2.

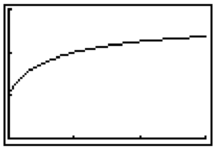


[−2, 2] by [−2, 6]

From the graph, the limit appears to be 5.

$$\lim_{x\to0}\frac{\sin 5x}{x} = \lim_{x\to0}\frac{5\cos 5x}{1} = 5$$

3.



[0, 3] by [0, 3]

From the graph, the limit appears to be 1. The limit leads to the indeterminate form ∞^0.

$$\ln\left(1+\frac{1}{x}\right)^x = x\ln\left(1+\frac{1}{x}\right) = \frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}}$$

$$\lim_{x\to0^+}\frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}} = \lim_{x\to0^+}\frac{\frac{1}{1+1/x}\left(-\frac{1}{x^2}\right)}{-\frac{1}{x^2}}$$

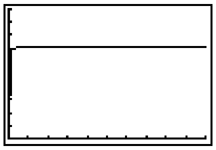
$$= \lim_{x\to0^+}\frac{1}{1+\frac{1}{x}}$$

$$= \lim_{x\to0^+}\frac{x}{x+1} = 0$$

Therefore,

$$\lim_{x\to0^+}\left(1+\frac{1}{x}\right)^x = \lim_{x\to0^+}f(x) = \lim_{x\to0^+}e^{\ln f(x)} = e^0 = 1.$$

4.

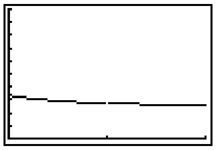


[0, 1000] by [0, 1]

From the graph, the limit appears to be about 0.714.

$$\lim_{x\to\infty}\frac{5x^2-3x}{7x^2+1} = \lim_{x\to\infty}\frac{10x-3}{14x} = \lim_{x\to\infty}\frac{10}{14} = \frac{5}{7} \approx 0.71429$$

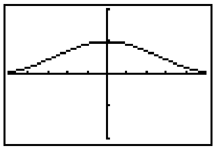
5. $\lim_{x\to1}\frac{x^3-1}{4x^3-x-3} = \lim_{x\to1}\frac{3x^2}{12x^2-1} = \frac{3}{11}$



[0, 2] by [0, 1]

The graph supports the answer.

6. $\lim_{x\to0}\frac{1-\cos x}{x^2} = \lim_{x\to0}\frac{\sin x}{2x} = \lim_{x\to0}\frac{\cos x}{2} = \frac{1}{2}$



[−5, 5] by [−1, 1]

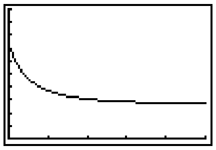
The graph supports the answer.

7. The limit leads to the indeterminate form 1^∞.

Let $\ln f(x) = \ln(e^x + x)^{1/x} = \dfrac{\ln(e^x + x)}{x}$.

$$\lim_{x\to 0^+} \frac{\ln(e^x + x)}{x} = \lim_{x\to 0^+} \frac{\frac{e^x + 1}{e^x + x}}{1} = \lim_{x\to 0^+} \frac{e^x + 1}{e^x + x} = \frac{2}{1} = 2$$

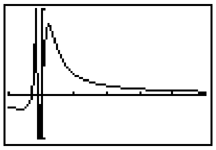
$$\lim_{x\to 0^+} (e^x + x)^{1/x} = \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} e^{\ln f(x)} = e^2$$



[0, 5] by [0, 10]

The graph supports the answer.

8. $\displaystyle\lim_{x\to\infty} \frac{2x^2 + 3x}{x^3 + x + 1} = \lim_{x\to\infty} \frac{4x + 3}{3x^2 + 1} = \lim_{x\to\infty} \frac{4}{6x} = 0$



[−5, 25] by [−1, 2]

The graph supports the answer.

9. (a)

x	10	10^2	10^3	10^4	10^5
$f(x)$	1.1513	0.2303	0.0345	0.00461	0.00058

Estimate the limit to be 0.

(b) $\displaystyle\lim_{x\to\infty} \frac{\ln x^5}{x} = \lim_{x\to\infty} \frac{5 \ln x}{x} = \lim_{x\to\infty} \frac{5/x}{1} = \frac{0}{1} = 0$

10. (a)

x	10^0	10^{-1}	10^{-2}	10^{-3}	10^{-4}
$f(x)$	0.1585	0.1666	0.1667	0.1667	0.1667

Estimate the limit to be $\dfrac{1}{6}$.

(b)
$$\begin{aligned}\lim_{x\to 0^+} \frac{x - \sin x}{x^3} &= \lim_{x\to 0^+} \frac{1 - \cos x}{3x^2}\\ &= \lim_{x\to 0^+} \frac{\sin x}{6x}\\ &= \lim_{x\to 0^+} \frac{\cos x}{6}\\ &= \frac{1}{6}\end{aligned}$$

11. Let $f(\theta) = \dfrac{\sin 3\theta}{\sin 4\theta}$.

θ	$\pm 10^0$	$\pm 10^{-1}$	$\pm 10^{-2}$	$\pm 10^{-3}$	$\pm 10^{-4}$
$f(\theta)$	−0.1865	0.7589	0.7501	0.7500	0.7500

Estimate the limit to be $\dfrac{3}{4}$.

$$\lim_{\theta\to 0} \frac{\sin 3\theta}{\sin 4\theta} = \lim_{\theta\to 0} \frac{3 \cos 3\theta}{4 \cos 4\theta} = \frac{3}{4}$$

12. Let $f(t) = \dfrac{1}{\sin t} - \dfrac{1}{t} = \dfrac{t - \sin t}{t \sin t}$.

t	$\pm 10^0$	$\pm 10^{-1}$	$\pm 10^{-2}$	$\pm 10^{-3}$
$f(t)$	±0.1884	±0.0167	±0.0017	±0.00017

Estimate the limit to be 0.

$$\begin{aligned}\lim_{t\to 0}\left(\frac{1}{\sin t} - \frac{1}{t}\right) &= \lim_{t\to 0} \frac{t - \sin t}{t \sin t}\\ &= \lim_{t\to 0} \frac{1 - \cos t}{t \cos t + \sin t}\\ &= \lim_{t\to 0} \frac{\sin t}{-t \sin t + \cos t + \cos t} = 0\end{aligned}$$

13. Let $f(x) = (1 + x)^{1/x}$.

x	10	10^2	10^3	10^4	10^5
$f(x)$	1.2710	1.0472	1.0069	1.0009	1.0001

Estimate the limit to be 1.

$$\ln f(x) = \frac{\ln(1 + x)}{x}$$

$$\lim_{x\to\infty} \frac{\ln(1 + x)}{x} = \lim_{x\to\infty} \frac{\frac{1}{1 + x}}{1} = \frac{0}{1} = 0$$

$$\lim_{x\to\infty} (1 + x)^{1/x} = \lim_{x\to\infty} f(x) = \lim_{x\to\infty} e^{\ln f(x)} = e^0 = 1$$

14. Let $f(x) = \dfrac{x - 2x^2}{3x^2 + 5x}$.

x	10	10^2	10^3	10^4	10^5
$f(x)$	−0.5429	−0.6525	−0.6652	−0.6665	−0.6667

Estimate the limit to be $-\dfrac{2}{3}$.

$$\lim_{x\to\infty} \frac{x - 2x^2}{3x^2 + 5x} = \lim_{x\to\infty} \frac{1 - 4x}{6x + 5} = \lim_{x\to\infty} -\frac{4}{6} = -\frac{2}{3}.$$

15. $\displaystyle\lim_{\theta\to 0} \frac{\sin \theta^2}{\theta} = \lim_{\theta\to 0} \frac{2\theta \cos \theta^2}{1} = (2)(0) \cos (0)^2 = 0$

16.
$$\begin{aligned}\lim_{\theta\to\pi/2} \frac{1 - \sin \theta}{1 + \cos 2\theta} &= \lim_{\theta\to\pi/2} \frac{-\cos \theta}{-2 \sin 2\theta}\\ &= \lim_{\theta\to\pi/2} \frac{\sin \theta}{-4 \cos 2\theta}\\ &= \frac{\sin \pi/2}{-4 \cos \pi}\\ &= \frac{1}{4}\end{aligned}$$

17. $\displaystyle\lim_{t\to 0} \frac{\cos t - 1}{e^t - t - 1} = \lim_{t\to 0} \frac{-\sin t}{e^t - 1} = \lim_{t\to 0} \frac{-\cos t}{e^t} = -1$

18. $$\lim_{t\to1}\frac{t-1}{\ln t-\sin\pi t}=\lim_{t\to1}\frac{1}{\frac{1}{t}-\pi\cos\pi t}$$
$$=\frac{1}{1-\pi(-1)}$$
$$=\frac{1}{\pi+1}$$

19. $$\lim_{x\to\infty}\frac{\ln(x+1)}{\log_2 x}=\lim_{x\to\infty}\frac{\frac{1}{x+1}}{\frac{1}{x\ln 2}}$$
$$=\lim_{x\to\infty}\frac{x\ln 2}{x+1}$$
$$=\lim_{x\to\infty}\ln 2$$
$$=\ln 2$$

20. $$\lim_{x\to\infty}\frac{\log_2 x}{\log_3(x+3)}=\lim_{x\to\infty}\frac{\frac{1}{x\ln 2}}{\frac{1}{(x+3)\ln 3}}$$
$$=\lim_{x\to\infty}\frac{(x+3)\ln 3}{x\ln 2}$$
$$=\lim_{x\to\infty}\frac{x\ln 3+3\ln 3}{x\ln 2}$$
$$=\lim_{x\to\infty}\frac{\ln 3}{\ln 2}$$
$$=\frac{\ln 3}{\ln 2}$$

21. $$\lim_{y\to0^+}\frac{\ln(y^2+2y)}{\ln y}=\lim_{y\to0^+}\frac{\frac{2y+2}{y^2+2y}}{\frac{1}{y}}$$
$$=\lim_{y\to0^+}\frac{y(2y+2)}{y^2+2y}$$
$$=\lim_{y\to0^+}\frac{2y^2+2y}{y^2+2y}$$
$$=\lim_{y\to0^+}\frac{4y+2}{2y+2}$$
$$=\frac{4(0)+2}{2(0)+2}=\frac{2}{2}=1$$

22. $$\lim_{y\to\pi/2}\left(\frac{\pi}{2}-y\right)\tan y=\lim_{y\to\pi/2}\frac{\left(\frac{\pi}{2}-y\right)\sin y}{\cos y}$$
$$=\lim_{y\to\pi/2}\frac{\left(\frac{\pi}{2}-y\right)\cos y+(-1)\sin y}{-\sin y}$$
$$=\frac{\left(\frac{\pi}{2}-\frac{\pi}{2}\right)\cos\frac{\pi}{2}+(-1)\sin\frac{\pi}{2}}{-\sin\frac{\pi}{2}}$$
$$=\frac{(-1)(1)}{-(1)}=1$$

23. $$\lim_{x\to0^+}x\ln x=\lim_{x\to0^+}\frac{\ln x}{\frac{1}{x}}$$
$$=\lim_{x\to0^+}\frac{\frac{1}{x}}{-\frac{1}{x^2}}$$
$$=\lim_{x\to0^+}\frac{-x^2}{x}$$
$$=\lim_{x\to0^+}-x=0$$

24. $$\lim_{x\to\infty}x\tan\frac{1}{x}=\lim_{x\to\infty}\frac{\tan\frac{1}{x}}{\frac{1}{x}}$$
$$=\lim_{x\to\infty}\frac{-\frac{1}{x^2}\sec^2\frac{1}{x}}{-\frac{1}{x^2}}$$
$$=\lim_{x\to\infty}\sec^2\frac{1}{x}$$
$$=\sec^2 0=1$$

25. $$\lim_{x\to0^+}(\csc x-\cot x+\cos x)$$
$$=\lim_{x\to0^+}\left(\frac{1}{\sin x}-\frac{\cos x}{\sin x}+\cos x\right)$$
$$=\lim_{x\to0^+}\frac{1-\cos x+\cos x\sin x}{\sin x}$$
$$=\lim_{x\to0^+}\frac{\sin x+\cos x\cos x-\sin x\sin x}{\cos x}=1$$

26. $\lim\limits_{x\to\infty} (\ln 2x - \ln(x+1)) = \lim\limits_{x\to\infty} \ln\left(\dfrac{2x}{x+1}\right)$

Let $f(x) = \dfrac{2x}{x+1}$.

$$\lim_{x\to\infty} \frac{2x}{x+1} = \lim_{x\to\infty} \frac{2}{1} = 2$$

Therefore,

$$\lim_{x\to\infty} (\ln 2x - \ln(x+1)) = \lim_{x\to\infty} \ln f(x) = \ln 2$$

27. $\lim\limits_{x\to0^+} (\ln x - \ln \sin x) = \lim\limits_{x\to0^+} \ln \dfrac{x}{\sin x}$

Let $f(x) = \dfrac{x}{\sin x}$.

$$\lim_{x\to0^+} \frac{x}{\sin x} = \lim_{x\to0^+} \frac{1}{\cos x} = 1$$

Therefore,

$$\lim_{x\to0^+} (\ln x - \ln \sin x) = \lim_{x\to0^+} \ln f(x) = \ln 1 = 0$$

28. $\lim\limits_{x\to0^+} \left(\dfrac{1}{x} - \dfrac{1}{\sqrt{x}}\right) = \lim\limits_{x\to0^+} \dfrac{1-\sqrt{x}}{x} = \infty$

29. The limit leads to the indeterminate form 1^∞.

Let $f(x) = (e^x + x)^{1/x}$.

$$\ln (e^x + x)^{1/x} = \frac{\ln(e^x + x)}{x}$$

$$\lim_{x\to0} \frac{\ln(e^x + x)}{x} = \lim_{x\to0} \frac{\dfrac{e^x+1}{e^x+x}}{1} = 2$$

$$\lim_{x\to0} (e^x + x)^{1/x} = \lim_{x\to0} e^{\ln f(x)} = e^2$$

30. The limit leads to the indeterminate form ∞^0.

Let $f(x) = \left(\dfrac{1}{x^2}\right)^x$.

$$\ln\left(\frac{1}{x^2}\right)^x = x \ln\left(\frac{1}{x^2}\right) = \frac{\ln\left(\dfrac{1}{x^2}\right)}{\dfrac{1}{x}}$$

$$\lim_{x\to0} \frac{\ln\left(\dfrac{1}{x^2}\right)}{\dfrac{1}{x}} = \lim_{x\to0} \frac{\dfrac{-2/x^3}{1/x^2}}{-1/x^2} = \lim_{x\to0} 2x = 0$$

$$\lim_{x\to0} \left(\frac{1}{x^2}\right)^x = \lim_{x\to0} e^{\ln f(x)} = e^0 = 1$$

31. $\lim\limits_{x\to\pm\infty} \dfrac{3x-5}{2x^2 - x + 2} = \lim\limits_{x\to\pm\infty} \dfrac{3}{4x-1} = 0$

32. $\lim\limits_{x\to0} \dfrac{\sin 7x}{\tan 11x} = \lim\limits_{x\to0} \dfrac{7\cos 7x}{11\sec^2 11x} = \dfrac{7}{11}$

33. The limit leads to the indeterminate form ∞^0.

Let $f(x) = (\ln x)^{1/x}$.

$$\ln(\ln x)^{1/x} = \frac{\ln(\ln x)}{x}$$

$$\lim_{x\to\infty} \frac{\ln(\ln x)}{x} = \lim_{x\to\infty} \frac{\dfrac{1/x}{\ln x}}{1} = \lim_{x\to\infty} \frac{1}{x \ln x} = 0$$

$$\lim_{x\to\infty} (\ln x)^{1/x} = \lim_{x\to\infty} e^{\ln f(x)} = e^0 = 1$$

34. The limit leads to the indeterminate form ∞^0.

Let $f(x) = (1+2x)^{1/(2\ln x)}$.

$$\ln(1+2x)^{1/(2\ln x)} = \frac{\ln(1+2x)}{2\ln x}$$

$$\lim_{x\to\infty} \frac{\ln(1+2x)}{2\ln x} = \lim_{x\to\infty} \frac{\dfrac{2}{1+2x}}{\dfrac{2}{x}} = \lim_{x\to\infty} \frac{x}{1+2x} = \lim_{x\to\infty} \frac{1}{2} = \frac{1}{2}$$

$$\lim_{x\to\infty} (1+2x)^{1/(2\ln x)} = \lim_{x\to\infty} e^{\ln f(x)} = e^{1/2} = \sqrt{e}$$

35. The limit leads to the indeterminate form 0^0.

Let $f(x) = (x^2 - 2x + 1)^{x-1}$

$$\ln(x^2 - 2x + 1)^{x-1} = (x-1)\ln(x^2 - 2x + 1)$$

$$= \frac{\ln(x^2 - 2x + 1)}{\dfrac{1}{x-1}}$$

$$\lim_{x\to1} \frac{\ln(x^2 - 2x + 1)}{\dfrac{1}{x-1}} = \lim_{x\to1} \frac{\dfrac{2x-2}{x^2-2x+1}}{-\dfrac{1}{(x-1)^2}}$$

$$= \lim_{x\to1} \frac{\dfrac{2(x-1)}{(x-1)^2}}{-\dfrac{1}{(x-1)^2}}$$

$$= \lim_{x\to1} -2(x-1) = 0$$

$$\lim_{x\to1} (x^2 - 2x + 1)^{x-1} = \lim_{x\to1} e^{\ln f(x)} = e^0 = 1$$

36. The limit leads to the indeterminate form 0^0.

Let $f(x) = (\cos x)^{\cos x}$.

$$\ln(\cos x)^{\cos x} = (\cos x)\ln(\cos x) = \frac{\ln(\cos x)}{\sec x}$$

$$\lim_{x\to\pi/2^-} \frac{\ln(\cos x)}{\sec x} = \lim_{x\to\pi/2^-} \frac{\dfrac{-\sin x}{\cos x}}{\sec x \tan x}$$

$$= \lim_{x\to\pi/2^-} \frac{-\tan x}{\sec x \tan x}$$

$$= \lim_{x\to\pi/2^-} -\cos x = 0$$

$$\lim_{x\to\pi/2^-} (\cos x)^{\cos x} = \lim_{x\to\pi/2^-} e^{\ln f(x)} = e^0 = 1$$

37. The limit leads to the indeterminate form 1^{∞}.

Let $f(x) = (1 + x)^{1/x}$.

$$\ln(1 + x)^{1/x} = \frac{\ln(1 + x)}{x}$$

$$\lim_{x\to0^+}\frac{\ln(1 + x)}{x} = \lim_{x\to0^+}\frac{\frac{1}{1 + x}}{1} = 1$$

$$\lim_{x\to0^+}(1 + x)^{1/x} = \lim_{x\to0^+} e^{\ln f(x)} = e^1 = e$$

38. The limit leads to the indeterminate form 1^{∞}.

Let $f(x) = x^{1/(x-1)}$.

$$\ln x^{1/(x-1)} = \frac{\ln x}{x - 1}$$

$$\lim_{x\to1}\frac{\ln x}{x - 1} = \lim_{x\to1}\frac{1/x}{1} = 1$$

$$\lim_{x\to1} x^{1/(x-1)} = \lim_{x\to1} e^{\ln f(x)} = e^1 = e$$

39. The limit leads to the indeterminate form 0^0.

Let $f(x) = (\sin x)^x$.

$$\ln(\sin x)^x = x\ln(\sin x) = \frac{\ln(\sin x)}{\frac{1}{x}}$$

$$\lim_{x\to0^+}\frac{\ln(\sin x)}{\frac{1}{x}} = \lim_{x\to0^+}\frac{\frac{\cos x}{\sin x}}{-\frac{1}{x^2}}$$

$$= \lim_{x\to0^+}\frac{-x^2\cos x}{\sin x}$$

$$= \lim_{x\to0^+}\frac{x^2\sin x - 2x\cos x}{\cos x} = 0$$

$$\lim_{x\to0^+}(\sin x)^x = \lim_{x\to0^+} e^{\ln f(x)} = e^0 = 1$$

40. The limit leads to the indeterminate form 0^0.

Let $f(x) = (\sin x)^{\tan x}$

$$\ln(\sin x)^{\tan x} = \tan x\ln(\sin x) = \frac{\ln(\sin x)}{\cot x}$$

$$\lim_{x\to0^+}\frac{\ln(\sin x)}{\cot x} = \lim_{x\to0^+}\frac{\frac{\cos x}{\sin x}}{-\csc^2 x} = \lim_{x\to0^+}(-\sin x\cos x) = 0$$

$$\lim_{x\to0^+}(\sin x)^{\tan x} = \lim_{x\to0^+} e^{\ln f(x)} = e^0 = 1$$

41. The limit leads to the indeterminate form $1^{-\infty}$.

Let $f(x) = x^{1/(1-x)}$.

$$\ln x^{1/(1-x)} = \frac{\ln x}{1 - x}$$

$$\lim_{x\to1^+}\frac{\ln x}{1 - x} = \lim_{x\to1^+}\frac{\frac{1}{x}}{-1} = -1$$

$$\lim_{x\to1^+} x^{1/(1-x)} = \lim_{x\to1^+} e^{\ln f(x)} = e^{-1} = \frac{1}{e}$$

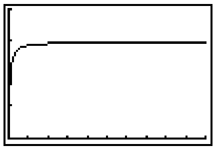
42. $$\int_x^{2x}\frac{dt}{t} = \Big[\ln|t|\Big]_x^{2x} = \ln|2x| - \ln|x| = \ln\left|\frac{2x}{x}\right|$$

$$\lim_{x\to\infty}\int_x^{2x}\frac{dt}{t} = \lim_{x\to\infty}\ln\left|\frac{2x}{x}\right| = \lim_{x\to\infty}\ln 2 = \ln 2$$

43. **(a)** L'Hôpital's Rule does not help because applying L'Hôpital's Rule to this quotient essentially "inverts" the problem by interchanging the numerator and denominator (see below). It is still essentially the same problem and one is no closer to a solution. Applying L'Hôpital's Rule a second time returns to the original problem.

$$\lim_{x\to\infty}\frac{\sqrt{9x + 1}}{\sqrt{x + 1}} = \lim_{x\to\infty}\frac{(9/2)(9x + 1)^{-1/2}}{(1/2)(x + 1)^{-1/2}} = \lim_{x\to\infty}\frac{9\sqrt{x + 1}}{\sqrt{9x + 1}}$$

(b)



[0, 100] by [0, 4]

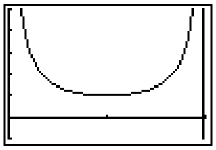
The limit appears to be 3.

(c) $$\lim_{x\to\infty}\frac{\sqrt{9x + 1}}{\sqrt{x + 1}} = \lim_{x\to\infty}\frac{\sqrt{9 + \frac{1}{x}}}{\sqrt{1 + \frac{1}{x}}} = \frac{\sqrt{9}}{\sqrt{1}} = 3$$

44. **(a)** L'Hôpital's Rule does not help because applying L'Hôpital's Rule to this quotient essentially "inverts" the problem by interchanging the numerator and denominator (see below). It is still essentially the same problem and one is no closer to a solution. Applying L'Hôpital's Rule a second time returns to the original problem.

$$\lim_{x\to\pi/2}\frac{\sec x}{\tan x} = \lim_{x\to\pi/2}\frac{\sec x\tan x}{\sec^2 x} = \lim_{x\to\pi/2}\frac{\tan x}{\sec x}$$

(b)



[0, π] by [−1, 5]

The limit appears to be 1.

(c) $$\lim_{x\to\pi/2}\frac{\sec x}{\tan x} = \lim_{x\to\pi/2}\frac{\frac{1}{\cos x}}{\frac{\sin x}{\cos x}} = \lim_{x\to\pi/2}\frac{1}{\sin x} = 1$$

45. Possible answers:

(a) $f(x) = 7(x - 3)$; $g(x) = x - 3$

$$\lim_{x\to3}\frac{f(x)}{g(x)} = \lim_{x\to3}\frac{7(x - 3)}{x - 3} = \lim_{x\to3}\frac{7}{1} = 7$$

(b) $f(x) = (x - 3)^2$; $g(x) = x - 3$

$$\lim_{x\to3}\frac{f(x)}{g(x)} = \lim_{x\to3}\frac{(x - 3)^2}{x - 3} = \lim_{x\to3}\frac{2(x - 3)}{1} = 0$$

(c) $f(x) = x - 3$; $g(x) = (x - 3)^3$

$$\lim_{x\to3}\frac{f(x)}{g(x)} = \lim_{x\to3}\frac{x - 3}{(x - 3)^3} = \lim_{x\to3}\frac{1}{3(x - 3)^2} = \infty$$

46. Answers may vary.

(a) $f(x) = 3x + 1;\ g(x) = x$

$$\lim_{x\to\infty}\frac{f(x)}{g(x)} = \lim_{x\to\infty}\frac{3x+1}{x} = \lim_{x\to\infty}\frac{3}{1} = 3$$

(b) $f(x) = x + 1;\ g(x) = x^2$

$$\lim_{x\to\infty}\frac{f(x)}{g(x)} = \lim_{x\to\infty}\frac{x+1}{x^2} = \lim_{x\to\infty}\frac{1}{2x} = 0$$

(c) $f(x) = x^2;\ g(x) = x + 1$

$$\lim_{x\to\infty}\frac{f(x)}{g(x)} = \lim_{x\to\infty}\frac{x^2}{x+1} = \lim_{x\to\infty}\frac{2x}{1} = \infty$$

47. Find c such that $\lim_{x\to0} f(x) = c$.

$$\begin{aligned}\lim_{x\to0} f(x) &= \lim_{x\to0}\frac{9x - 3\sin 3x}{5x^3}\\ &= \lim_{x\to0}\frac{9 - 9\cos 3x}{15x^2}\\ &= \lim_{x\to0}\frac{27\sin 3x}{30x}\\ &= \lim_{x\to0}\frac{81\cos 3x}{30} = \frac{81}{30} = \frac{27}{10}\end{aligned}$$

Thus, $c = \frac{27}{10}$. This works since $\lim_{x\to0} f(x) = c = f(0)$, so f is continuous.

48. $f(x)$ is defined at $x \neq 0$. $\lim_{x\to0} f(x)$ leads to the indeterminate form 0^0.

$$\ln|x|^x = x\ln|x| = \frac{\ln|x|}{\frac{1}{x}}$$

$$\lim_{x\to0}\frac{\ln|x|}{\frac{1}{x}} = \lim_{x\to0}\frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x\to0} -x = 0$$

$$\lim_{x\to0}|x|^x = \lim_{x\to0} e^{x\ln|x|} = e^0 = 1$$

Thus, f has a removable discontinuity at $x = 0$. Extend the definition of f by letting $f(0) = 1$.

49. (a) The limit leads to the indeterminate form 1^∞.

Let $f(k) = \left(1 + \frac{r}{k}\right)^{kt}$.

$$\ln f(k) = kt\ln\left(1 + \frac{r}{k}\right) = \frac{t\ln\left(1+\frac{r}{k}\right)}{\frac{1}{k}}$$

$$\begin{aligned}\lim_{k\to\infty}\frac{t\ln\left(1+\frac{r}{k}\right)}{\frac{1}{k}} &= \lim_{k\to\infty}\frac{t\left(-\frac{r}{k^2}\right)\left(1+\frac{r}{k}\right)^{-1}}{-\frac{1}{k^2}}\\ &= \lim_{k\to\infty}\frac{rt}{1+\frac{r}{k}} = \frac{rt}{1} = rt\end{aligned}$$

$$\begin{aligned}\lim_{k\to\infty} A_0\left(1+\frac{r}{k}\right)^{kt} &= A_0\lim_{k\to\infty}\left(1+\frac{r}{k}\right)^{kt}\\ &= A_0\lim_{k\to\infty} e^{\ln f(k)}\\ &= A_0 e^{rt}\end{aligned}$$

(b) Part (a) shows that as the number of compoundings per year increases toward infinity, the limit of interest compounded k times per year is interest compounded continuously.

50. (a) For $x \neq 0$, $\frac{f'(x)}{g'(x)} = \frac{1}{1} = 1$.

$$\lim_{x\to0}\frac{f'(x)}{g'(x)} = 1$$

$$\lim_{x\to0}\frac{f(x)}{g(x)} = \frac{2}{1} = 2$$

(b) This does not contradict L'Hôpital's Rule since $\lim_{x\to0} f(x) = 2$ and $\lim_{x\to0} g(x) = 1$.

51. (a) $A(t) = \int_0^t e^{-x}\,dx = \Big[-e^{-x}\Big]_0^t = -e^{-t} + 1$

$$\lim_{t\to\infty} A(t) = \lim_{t\to\infty}(-e^{-t} + 1) = \lim_{t\to\infty}\left(-\frac{1}{e^t} + 1\right) = 1$$

(b)

$$\begin{aligned}V(t) &= \pi\int_0^t (e^{-x})^2\,dx\\ &= \pi\int_0^t e^{-2x}\,dx\\ &= \pi\left[-\frac{1}{2}e^{-2x}\right]_0^t\\ &= \pi\left(-\frac{1}{2}e^{-2t} + \frac{1}{2}\right)\\ &= \frac{\pi}{2}(-e^{-2t} + 1)\end{aligned}$$

$$\lim_{t\to\infty}\frac{V(t)}{A(t)} = \lim_{t\to\infty}\frac{\frac{\pi}{2}(-e^{-2t}+1)}{-e^{-t}+1} = \frac{\frac{\pi}{2}(1)}{1} = \frac{\pi}{2}$$

(c)

$$\begin{aligned}\lim_{t\to0^+}\frac{V(t)}{A(t)} &= \lim_{t\to0^+}\frac{\frac{\pi}{2}(-e^{-2t}+1)}{-e^{-t}+1}\\ &= \lim_{t\to0^+}\frac{\frac{\pi}{2}(2e^{-2t})}{e^{-t}}\\ &= \frac{\frac{\pi}{2}(2)}{1} = \pi\end{aligned}$$

52. (a)

x	$f(x)$
0.1	0.04542
0.01	0.00495
0.001	0.00050
0.0001	0.00005

The limit appears to be 0.

(b) $\lim_{x\to0}\frac{\sin x}{1+2x} = \frac{0}{1} = 0$

L'Hôpital's Rule is not applied here because the limit is not of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$, since the denominator has limit 1.

53. **(a)** $f(x) = e^{x \ln(1 + 1/x)}$

$1 + \frac{1}{x} > 0$ when $x < -1$ or $x > 0$

Domain: $(-\infty, -1) \cup (0, \infty)$

(b) The form is 0^{-1}, so $\lim_{x\to -1^-} f(x) = \infty$

(c)
$$\lim_{x\to-\infty} x \ln\left(1 + \frac{1}{x}\right) = \lim_{x\to-\infty} \frac{\ln\left(1 + \frac{1}{x}\right)}{\frac{1}{x}}$$
$$= \lim_{x\to-\infty} \frac{\left(-\frac{1}{x^2}\right)\left(1 + \frac{1}{x}\right)^{-1}}{-\frac{1}{x^2}}$$
$$= \lim_{x\to-\infty} \frac{1}{1 + \frac{1}{x}} = 1$$
$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} e^{x \ln(1+1/x)} = e$$

54. **(a)** Because the difference in the numerator is so small compared to the values being subtracted, any calculator or computer with limited precision will give the incorrect result that $1 - \cos x^6$ is 0 for even moderately small values of x. For example, at $x = 0.1$, $\cos x^6 \approx 0.9999999999995$ (13 places), so on a 10-place calculator, $\cos x^6 = 1$ and $1 - \cos x^6 = 0$.

(b) Same reason as in part (a) applies.

(c)
$$\lim_{x\to 0} \frac{1 - \cos x^6}{x^{12}} = \lim_{x\to 0} \frac{6x^5 \sin x^6}{12x^{11}}$$
$$= \lim_{x\to 0} \frac{\sin x^6}{2x^6}$$
$$= \lim_{x\to 0} \frac{6x^5 \cos x^6}{12x^5}$$
$$= \lim_{x\to 0} \frac{\cos x^6}{2} = \frac{1}{2}$$

(d) The graph and/or table on a grapher show the value of the function to be 0 for x-values moderately close to 0, but the limit is 1/2. The calculator is giving unreliable information because there is significant round-off error in computing values of this function on a limited precision device.

55. **(a)** $f'(x) = 3x^2$, $g'(x) = 2x - 1$

$f(1) - f(-1) = 2$, $g(1) - g(-1) = -2$

$$\frac{3c^2}{2c - 1} = \frac{2}{-2}$$
$$3c^2 = -2c + 1$$
$$3c^2 + 2c - 1 = 0$$
$$(3c - 1)(c + 1) = 0$$
$$c = \frac{1}{3} \text{ or } c = -1$$

The value of c that satisfies the property is $c = \frac{1}{3}$.

(b) $f'(x) = -\sin x$, $g'(x) = \cos x$

$f\left(\frac{\pi}{2}\right) - f(0) = -1$, $g\left(\frac{\pi}{2}\right) - g(0) = 1$

$$\frac{-\sin c}{\cos c} = \frac{-1}{1}$$
$$\tan c = 1$$
$$c = \tan^{-1} 1 = \frac{\pi}{4} \text{ on } \left(0, \frac{\pi}{2}\right)$$

56. **(a)** $\ln f(x)^{g(x)} = g(x) \ln f(x)$

$$\lim_{x\to c} (g(x) \ln f(x)) = \left(\lim_{x\to c} g(x)\right)\left(\lim_{x\to c} \ln f(x)\right)$$
$$= \infty(-\infty) = -\infty$$
$$\lim_{x\to c} f(x)^{g(x)} = \lim_{x\to c} e^{\ln f(x)^{g(x)}} = e^{-\infty} = 0$$

(b)
$$\lim_{x\to c} (g(x) \ln f(x)) = \left(\lim_{x\to c} g(x)\right)\left(\lim_{x\to c} \ln f(x)\right)$$
$$= (-\infty)(-\infty) = \infty$$
$$\lim_{x\to c} f(x)^{g(x)} = \lim_{x\to c} e^{\ln f(x)^{g(x)}} = e^{\infty} = \infty$$

■ Section 8.2 Relative Rates of Growth
(pp. 425–433)

Exploration 1 Comparing Rates of Growth as $x\to\infty$

1. $\lim_{x\to\infty} \frac{a^x}{x^2} = \lim_{x\to\infty} \frac{(\ln a)(a^x)}{2x} = \lim_{x\to\infty} \frac{(\ln a)^2 a^x}{2} = \infty$, so a^x grows faster than x^2 as $x\to\infty$.

2. $\lim_{x\to\infty} \frac{3^x}{2^x} = \lim_{x\to\infty} 1.5^x = \infty$

3. $\lim_{x\to\infty} \frac{a^x}{b^x} = \lim_{x\to\infty} \left(\frac{a}{b}\right)^x = \infty$ because $\frac{a}{b} > 1$.

Quick Review 8.2

1. $\lim_{x\to\infty} \frac{\ln x}{e^x} = \lim_{x\to\infty} \frac{\frac{1}{x}}{e^x} = 0$

2. $\lim_{x\to\infty} \frac{e^x}{x^3} = \lim_{x\to\infty} \frac{e^x}{3x^2} = \lim_{x\to\infty} \frac{e^x}{6x} = \lim_{x\to\infty} \frac{e^x}{6} = \infty$

3. $\lim_{x\to-\infty} \frac{x^2}{e^{2x}} = \infty$

4. $\lim_{x\to\infty} \frac{x^2}{e^{2x}} = \lim_{x\to\infty} \frac{2x}{2e^{2x}} = \lim_{x\to\infty} \frac{2}{4e^{2x}} = 0$

5. $-3x^4$

6. $\frac{2x^3}{x} = 2x^2$

7. $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{x + \ln x}{x} = \lim_{x\to\infty} \frac{1 + \frac{1}{x}}{1} = 1$

8. $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{\sqrt{4x^2 + 5x}}{2x} = \lim_{x\to\infty} \sqrt{1 + \frac{5}{4x}} = 1$

9. (a) $f(x) = \dfrac{e^x + x^2}{e^x} = 1 + \dfrac{x^2}{e^x}$

$f'(x) = \dfrac{2xe^x - x^2e^x}{e^{2x}} = \dfrac{2x - x^2}{e^x}$

$\dfrac{2x - x^2}{e^x} = 0$

$x(2 - x) = 0$

$x = 0$ or $x = 2$

$f'(x) < 0$ for $x < 0$ or $x > 2$

The graph decreases, increases, and then decreases.

$f(0) = 1$; $f(2) = 1 + \dfrac{4}{e^2} \approx 1.541$

f has a local maximum at $\approx (2, 1.541)$ and has a local minimum at $(0, 1)$.

(b) f is increasing on $[0, 2]$

(c) f is decreasing on $(-\infty, 0]$ and $[2, \infty)$.

10. $f(x) = \dfrac{x + \sin x}{x} = 1 + \dfrac{\sin x}{x}, x \neq 0$

Observe that $\left|\dfrac{\sin x}{x}\right| < 1$ since $|\sin x| < |x|$ for $x \neq 0$.

$\lim\limits_{x\to 0} f(x) = 1 + \lim\limits_{x\to 0} \dfrac{\sin x}{x} = 1 + 1 = 2$

Thus the values of f get close to 2 as x gets close to 0, so f doesn't have an absolute maximum value. f is not defined at 0.

Section 8.2 Exercises

1. $\lim\limits_{x\to\infty} \dfrac{x^3 - 3x + 1}{e^x} = \lim\limits_{x\to\infty} \dfrac{3x^2 - 3}{e^x} = \lim\limits_{x\to\infty} \dfrac{6x}{e^x} = \lim\limits_{x\to\infty} \dfrac{6}{e^x} = 0$

$x^3 - 3x + 1$ grows slower than e^x as $x\to\infty$.

2. First observe that $\sqrt{1 + x^4}$ grows at the same rate as x^2.

$\lim\limits_{x\to\infty} \dfrac{\sqrt{1 + x^4}}{x^2} = \lim\limits_{x\to\infty} \sqrt{\dfrac{1 + x^4}{x^4}} = \lim\limits_{x\to\infty} \sqrt{\dfrac{1}{x^4} + 1} = 1$

Next compare x^2 with e^x.

$\lim\limits_{x\to\infty} \dfrac{x^2}{e^x} = \lim\limits_{x\to\infty} \dfrac{2x}{e^x} = \lim\limits_{x\to\infty} \dfrac{2}{e^x} = 0$

x^2 grows slower than e^x as $x\to\infty$, so $\sqrt{1 + x^4}$ grows slower than e^x as $x\to\infty$.

3. $\lim\limits_{x\to\infty} \dfrac{4^x}{e^x} = \lim\limits_{x\to\infty} \left(\dfrac{4}{e}\right)^x = \infty$ since $\dfrac{4}{e} > 1$.

4^x grows faster than e^x as $x\to\infty$.

4. $\lim\limits_{x\to\infty} \dfrac{(5/2)^x}{e^x} = \lim\limits_{x\to\infty} \left(\dfrac{5}{2e}\right)^x = 0$ since $\dfrac{5}{2e} < 1$.

$\left(\dfrac{5}{2}\right)^x$ grows slower than e^x as $x\to\infty$.

5. $\lim\limits_{x\to\infty} \dfrac{e^{x+1}}{e^x} = \lim\limits_{x\to\infty} e = e$

e^{x+1} grows at the same rate as e^x as $x\to\infty$.

6. $\lim\limits_{x\to\infty} \dfrac{x \ln x - x}{e^x} = \lim\limits_{x\to\infty} \dfrac{x\left(\dfrac{1}{x}\right) + \ln x - 1}{e^x}$

$= \lim\limits_{x\to\infty} \dfrac{\ln x}{e^x}$

$= \lim\limits_{x\to\infty} \dfrac{1/x}{e^x} = 0$

$x \ln x - x$ grows slower than e^x as $x\to\infty$.

7. $\lim\limits_{x\to\infty} \dfrac{e^{\cos x}}{e^x} = 0$ since $e^{\cos x} \leq e$ for all x.

$e^{\cos x}$ grows slower than e^x as $x\to\infty$.

8. $\lim\limits_{x\to\infty} \dfrac{xe^x}{e^x} = \lim\limits_{x\to\infty} x = \infty$

xe^xgrows faster than e^x as $x\to\infty$.

9. $\lim\limits_{x\to\infty} \dfrac{x^{1000}}{e^x} = 0$ $\Big($Repeated application of L'Hôpital's Rule gets $\lim\limits_{x\to\infty} \dfrac{1000!}{e^x} = 0.\Big)$ x^{1000} grows slower than e^x as $x\to\infty$.

10. $\lim\limits_{x\to\infty} \dfrac{(e^x + e^{-x})/2}{e^x} = \lim\limits_{x\to\infty} \left(\dfrac{1}{2} + \dfrac{1}{2e^{2x}}\right) = \dfrac{1}{2}$

$\dfrac{e^x + e^{-x}}{2}$ grows at the same rate as e^x as $x\to\infty$.

11. $\lim\limits_{x\to\infty} \dfrac{x^2 + 4x}{x^2} = \lim\limits_{x\to\infty} \left(1 + \dfrac{4}{x}\right) = 1$

$x^2 + 4x$ grows at the same rate as x^2 as $x\to\infty$.

12. $\lim\limits_{x\to\infty} \dfrac{x^3 + 3}{x^2} = \lim\limits_{x\to\infty} \left(x + \dfrac{3}{x^2}\right) = \infty$

$x^3 + 3$ grows faster than x^2 as $x\to\infty$.

13. $\lim\limits_{x\to\infty} \dfrac{15x + 3}{x^2} = \lim\limits_{x\to\infty} \left(\dfrac{15}{x} + \dfrac{3}{x^2}\right) = 0$

$15x + 3$ grows slower than x^2 as $x\to\infty$.

14. $\lim\limits_{x\to\infty} \dfrac{\sqrt{x^4 + 5x}}{x^2} = \lim\limits_{x\to\infty} \sqrt{\dfrac{x^4 + 5x}{x^4}} = \lim\limits_{x\to\infty} \sqrt{1 + \dfrac{5}{x^3}} = 1$

$\sqrt{x^4 + 5x}$ grows at the same rate as x^2 as $x\to\infty$.

15. $\lim\limits_{x\to\infty} \dfrac{\ln x}{x^2} = \lim\limits_{x\to\infty} \dfrac{1/x}{2x} = \lim\limits_{x\to\infty} \dfrac{1}{2x^2} = 0$

$\ln x$ grows slower than x^2 as $x\to\infty$.

16. $\lim\limits_{x\to\infty} \dfrac{2^x}{x^2} = \lim\limits_{x\to\infty} \dfrac{(\ln 2)2^x}{2x} = \lim\limits_{x\to\infty} \dfrac{(\ln 2)^2\, 2^x}{2} = \infty.$

2^x grows faster than x^2 as $x\to\infty$.

17. $\lim\limits_{x\to\infty} \dfrac{\log_2 x^2}{\ln x} = \lim\limits_{x\to\infty} \dfrac{2 \log_2 x}{\ln x} = \lim\limits_{x\to\infty} \dfrac{2(\ln x)/(\ln 2)}{\ln x} = \dfrac{2}{\ln 2}$

$\log_2 x^2$ grows at the same rate as $\ln x$ as $x\to\infty$.

18. $\lim\limits_{x\to\infty} \dfrac{\log \sqrt{x}}{\ln x} = \lim\limits_{x\to\infty} \dfrac{\log x}{2 \ln x} = \lim\limits_{x\to\infty} \dfrac{(\ln x)/(\ln 10)}{2 \ln x} = \dfrac{1}{2 \ln 10}$

$\log \sqrt{x}$ grows at the same rate as $\ln x$ as $x\to\infty$.

19. $\lim\limits_{x\to\infty} \dfrac{1/\sqrt{x}}{\ln x} = \lim\limits_{x\to\infty} \dfrac{1}{\sqrt{x}\ln x} = 0$

$\dfrac{1}{\sqrt{x}}$ grows slower than $\ln x$ as $x\to\infty$.

20. $\lim\limits_{x\to\infty} \dfrac{e^{-x}}{\ln x} = \lim\limits_{x\to\infty} \dfrac{1}{e^{x}\ln x} = 0$

e^{-x} grows slower than $\ln x$ as $x\to\infty$.

21. $\lim\limits_{x\to\infty} \dfrac{x - 2\ln x}{\ln x} = \lim\limits_{x\to\infty} \left(\dfrac{x}{\ln x} - 2\right) = \lim\limits_{x\to\infty} \left(\dfrac{1}{1/x} - 2\right)$

$= \lim\limits_{x\to\infty} (x - 2) = \infty$

$x - 2\ln x$ grows faster than $\ln x$ as $x\to\infty$.

22. $\lim\limits_{x\to\infty} \dfrac{5\ln x}{\ln x} = 5$

$5\ln x$ grows at the same rate as $\ln x$ as $x\to\infty$.

23. Compare e^x to x^x.

$\lim\limits_{x\to\infty} \dfrac{e^x}{x^x} = \lim\limits_{x\to\infty} \left(\dfrac{e}{x}\right)^x = 0$

e^x grows slower than x^x.

Compare e^x to $(\ln x)^x$.

$\lim\limits_{x\to\infty} \dfrac{e^x}{(\ln x)^x} = \lim\limits_{x\to\infty} \left(\dfrac{e}{\ln x}\right)^x = 0$

e^x grows slower than $(\ln x)^x$.

Compare e^x to $e^{x/2}$.

$\lim\limits_{x\to\infty} \dfrac{e^x}{e^{x/2}} = \lim\limits_{x\to\infty} e^{x/2} = \infty$

e^x grows faster than $e^{x/2}$.

Compare x^x to $(\ln x)^x$.

$\lim\limits_{x\to\infty} \dfrac{x^x}{(\ln x)^x} = \lim\limits_{x\to\infty} \left(\dfrac{x}{\ln x}\right)^x = \infty$ since $\lim\limits_{x\to\infty} \dfrac{x}{\ln x} = \lim\limits_{x\to\infty} \dfrac{1}{1/x} = \infty.$

x^x grows faster than $(\ln x)^x$.

Thus, in order from slowest-growing to fastest-growing, we get $e^{x/2}$, e^x, $(\ln x)^x$, x^x.

24. Compare 2^x to x^2.

$\lim\limits_{x\to\infty} \dfrac{2^x}{x^2} = \lim\limits_{x\to\infty} \dfrac{(\ln 2)2^x}{2x} = \lim\limits_{x\to\infty} \dfrac{(\ln 2)^2\, 2^x}{2} = \infty$

2^x grows faster than x^2.

Compare 2^x to $(\ln 2)^x$.

$\lim\limits_{x\to\infty} \dfrac{2^x}{(\ln 2)^x} = \lim\limits_{x\to\infty} \left(\dfrac{2}{\ln 2}\right)^x = \infty$ since $\dfrac{2}{\ln 2} > 1.$

2^x grows faster than $(\ln 2)^x$.

Compare 2^x to e^x.

$\lim\limits_{x\to\infty} \dfrac{2^x}{e^x} = \lim\limits_{x\to\infty} \left(\dfrac{2}{e}\right)^x = 0$ since $\dfrac{2}{e} < 1.$

2^x grows slower than e^x.

Compare x^2 to $(\ln 2)^x$.

$\lim\limits_{x\to\infty} \dfrac{x^2}{(\ln 2)^x} = \infty$ since $\lim\limits_{x\to\infty} x^2 = \infty$ and $\lim\limits_{x\to\infty} (\ln 2)^x = 0.$

x^2 grows faster than $(\ln 2)^x$.

Thus, in order from slowest-growing to fastest-growing, we get $(\ln 2)^x$, x^2, 2^x, e^x.

25. Compare f_1 to f_2.

$\lim\limits_{x\to\infty} \dfrac{f_2(x)}{f_1(x)} = \lim\limits_{x\to\infty} \dfrac{\sqrt{10x+1}}{\sqrt{x}} = \lim\limits_{x\to\infty} \sqrt{10 + \dfrac{1}{x}} = \sqrt{10}$

Thus f_1 and f_2 grow at the same rate.

Compare f_1 to f_3.

$\lim\limits_{x\to\infty} \dfrac{f_3(x)}{f_1(x)} = \lim\limits_{x\to\infty} \dfrac{\sqrt{x+1}}{\sqrt{x}} = \lim\limits_{x\to\infty} \sqrt{1 + \dfrac{1}{x}} = 1$

Thus f_1 and f_3 grow at the same rate.

By transitivity, f_2 and f_3 grow at the same rate, so all three functions grow at the same rate as $x\to\infty$.

26. Compare f_1 to f_2.

$\lim\limits_{x\to\infty} \dfrac{f_2(x)}{f_1(x)} = \lim\limits_{x\to\infty} \dfrac{\sqrt{x^4+x}}{x^2} = \lim\limits_{x\to\infty} \sqrt{1 + \dfrac{1}{x^3}} = 1$

Thus f_1 and f_2 grow at the same rate.

Compare f_1 to f_3.

$\lim\limits_{x\to\infty} \dfrac{f_3(x)}{f_1(x)} = \lim\limits_{x\to\infty} \dfrac{\sqrt{x^4-x^3}}{x^2} = \lim\limits_{x\to\infty} \sqrt{1 - \dfrac{1}{x}} = 1$

Thus f_1 and f_3 grow at the same rate.

By transitivity, f_2 and f_3 grow at the same rate, so all three functions grow at the same rate as $x\to\infty$.

27. Compare f_1 to f_2.

$$\begin{aligned}\lim_{x\to\infty} \frac{f_2(x)}{f_1(x)} &= \lim_{x\to\infty} \frac{\sqrt{9^x+2^x}}{3^x}\\ &= \lim_{x\to\infty} \frac{\sqrt{9^x+2^x}}{\sqrt{9^x}}\\ &= \lim_{x\to\infty} \sqrt{1+\left(\frac{2}{9}\right)^x} = 1\end{aligned}$$

Thus f_1 and f_2 grow at the same rate.

Compare f_1 to f_3.

$$\begin{aligned}\lim_{x\to\infty} \frac{f_3(x)}{f_1(x)} &= \lim_{x\to\infty} \frac{\sqrt{9^x-4^x}}{3^x}\\ &= \lim_{x\to\infty} \frac{\sqrt{9^x-4^x}}{\sqrt{9^x}}\\ &= \lim_{x\to\infty} \sqrt{1-\left(\frac{4}{9}\right)^x} = 1\end{aligned}$$

Thus f_1 and f_3 grow at the same rate.

By transitivity, f_2 and f_3 grow at the same rate, so all three functions grow at the same rate as $x\to\infty$.

28. Compare f_1 to f_2.

$$\lim_{x\to\infty}\frac{f_2(x)}{f_1(x)} = \lim_{x\to\infty}\frac{\dfrac{x^4+2x^2-1}{x+1}}{x^3}$$

$$= \lim_{x\to\infty}\frac{x^4+2x^2-1}{x^4+x^3}$$

$$= \lim_{x\to\infty}\frac{1+\dfrac{2}{x^2}-\dfrac{1}{x^4}}{1+\dfrac{1}{x}} = 1$$

Thus f_1 and f_2 grow at the same rate.

Compare f_1 and f_3.

$$\lim_{x\to\infty}\frac{f_3(x)}{f_1(x)} = \lim_{x\to\infty}\frac{\dfrac{2x^5-1}{x^2+1}}{x^3}$$

$$= \lim_{x\to\infty}\frac{2x^5-1}{x^5+x^3}$$

$$= \lim_{x\to\infty}\frac{2-\dfrac{1}{x^5}}{1+\dfrac{1}{x^2}} = 2$$

Thus f_1 and f_3 grow at the same rate.
By transitivity, f_2 and f_3 grow at the same rate, so all three functions grow at the same rate.

29. **(a)** False, since $\lim\limits_{x\to\infty}\dfrac{x}{x} = 1 \neq 0$.

(b) False, since $\lim\limits_{x\to\infty}\dfrac{x}{x+5} = 1 \neq 0$.

(c) True, since $\lim\limits_{x\to\infty}\dfrac{x}{x+5} = 1 \leq 1$.

(d) True, since $\lim\limits_{x\to\infty}\dfrac{x}{2x} = \dfrac{1}{2} \leq 1$.

(e) True, since $\lim\limits_{x\to\infty}\dfrac{e^x}{e^{2x}} = \lim\limits_{x\to\infty}\dfrac{1}{e^x} = 0$.

(f) True, since $\lim\limits_{x\to\infty}\dfrac{x+\ln x}{x} = \lim\limits_{x\to\infty}\dfrac{1+\dfrac{1}{x}}{1} = 1 \leq 1$.

(g) False, since $\lim\limits_{x\to\infty}\dfrac{\ln x}{\ln 2x} = \lim\limits_{x\to\infty}\dfrac{1/x}{1/x} = 1 \neq 0$.

(h) True, since $\lim\limits_{x\to\infty}\dfrac{\sqrt{x^2+5}}{x} = \lim\limits_{x\to\infty}\sqrt{1+\dfrac{5}{x^2}} = 1 \leq 1$.

30. **(a)** True, since $\lim\limits_{x\to\infty}\dfrac{\dfrac{1}{x+3}}{\dfrac{1}{x}} = \lim\limits_{x\to\infty}\dfrac{x}{x+3} = 1 \leq 1$.

(b) True, since $\lim\limits_{x\to\infty}\dfrac{\dfrac{1}{x}+\dfrac{1}{x^2}}{\dfrac{1}{x}} = \lim\limits_{x\to\infty}\dfrac{1+\dfrac{1}{x}}{1} = 1 \leq 1$.

(c) False, since $\lim\limits_{x\to\infty}\dfrac{\dfrac{1}{x}-\dfrac{1}{x^2}}{\dfrac{1}{x}} = \lim\limits_{x\to\infty}\dfrac{1-\dfrac{1}{x}}{1} = 1 \neq 0$.

(d) True, since $\lim\limits_{x\to\infty}\dfrac{2+\cos x}{2} \leq \lim\limits_{x\to\infty}\dfrac{3}{2} = \dfrac{3}{2}$.

(e) True, since $\lim\limits_{x\to\infty}\dfrac{e^x+x}{e^x} = \lim\limits_{x\to\infty}\dfrac{e^x+1}{e^x}$

$$= \lim_{x\to\infty}\frac{1+\dfrac{1}{e^x}}{1} = 1 \leq 1.$$

(f) True, since $\lim\limits_{x\to\infty}\dfrac{x\ln x}{x^2} = \lim\limits_{x\to\infty}\dfrac{\ln x}{x} = \lim\limits_{x\to\infty}\dfrac{1/x}{1} = 0$.

(g) True, since $\lim\limits_{x\to\infty}\dfrac{\ln(\ln x)}{\ln x} = \lim\limits_{x\to\infty}\dfrac{\dfrac{1}{x\ln x}}{\dfrac{1}{x}}$

$$= \lim_{x\to\infty}\frac{1}{\ln x} = 0 \leq 1.$$

(h) False, since $\lim\limits_{x\to\infty}\dfrac{\ln x}{\ln(x^2+1)} = \lim\limits_{x\to\infty}\dfrac{1/x}{\dfrac{2x}{x^2+1}}$

$$= \lim_{x\to\infty}\frac{x^2+1}{2x^2} = \frac{1}{2} \neq 0.$$

31. From the graph, $\lim\limits_{x\to\infty}\dfrac{f(x)}{g(x)} = \infty$, so $\lim\limits_{x\to\infty}\dfrac{g(x)}{f(x)} = 0$.
Thus $g = o(f)$, so **ii** is true.

32. From the graph, $\lim\limits_{x\to\infty}\dfrac{f(x)}{g(x)} = 0$. Thus $f = o(g)$, so **i** is true.

33. From the graph, $\lim\limits_{x\to\infty}\dfrac{f(x)}{g(x)} \leq 1$ and not equal to zero. Thus, f and g grow at the same rate, so **iii** is true.

34. From the graph, $\lim\limits_{x\to\infty}\dfrac{f(x)}{g(x)} \leq 3$ and not equal to zero. Thus, f and g grow at the same rate, so **iii** is true.

35. **(a)** The nth derivative of x^n is $n!$, a constant. We can apply L'Hôpital's Rule n times to find $\lim\limits_{x\to\infty}\dfrac{e^x}{x^n}$.

$$\lim_{x\to\infty}\frac{e^x}{x^n} = \cdots = \lim_{x\to\infty}\frac{e^x}{n!} = \infty$$

Thus e^x grows faster than x^n as $x\to\infty$ for any positive integer n.

(b) The nth derivative of a^x, $a > 1$, is $(\ln a)^n a^x$. We can apply L'Hôpital's Rule n times to find $\lim\limits_{x\to\infty}\dfrac{a^x}{x^n}$.

$$\lim_{x\to\infty}\frac{a^x}{x^n} = \cdots = \lim_{x\to\infty}\frac{(\ln a)^n a^x}{n!} = \infty$$

Thus a^x grows faster than x^n as $x\to\infty$ for any positive integer n.

36. **(a)** Apply L'Hôpital's Rule n times to find

$$\lim_{x\to\infty} \frac{e^x}{a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0}.$$

$$\lim_{x\to\infty} \frac{e^x}{a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0} = \lim_{x\to\infty} \frac{e^x}{a_n n!} = \infty$$

Thus e^x grows faster than

$a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$ as $x\to\infty$.

(b) Apply L'Hôpital's Rule n times to find

$$\lim_{x\to\infty} \frac{a^x}{a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0}.$$

$$\lim_{x\to\infty} \frac{a^x}{a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0} = \cdots$$

$$= \lim_{x\to\infty} \frac{(\ln a)^n a^x}{a_n n!} = \infty$$

Thus a_x grows faster than

$a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$ as $x\to\infty$.

37. **(a)** $$\lim_{x\to\infty} \frac{\ln x}{x^{1/n}} = \lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{1}{n}x^{(1/n)-1}} = \lim_{x\to\infty} \frac{n}{x^{1/n}} = 0$$

Thus $\ln x$ grows slower than $x^{1/n}$ as $x\to\infty$ for any

positive integer n.

(b) $$\lim_{x\to\infty} \frac{\ln x}{x^a} = \lim_{x\to\infty} \frac{\frac{1}{x}}{ax^{a-1}} = \lim_{x\to\infty} \frac{1}{ax^a} = 0$$

Thus $\ln x$ grows slower than x^a as $x\to\infty$ for any number

$a > 0$.

38. $$\lim_{x\to\infty} \frac{\ln x}{a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0}$$

$$= \lim_{x\to\infty} \frac{\frac{1}{x}}{na_nx^{n-1} + (n-1)a_{n-1}x^{n-2} + \cdots + a_1}$$

$$= \lim_{x\to\infty} \frac{1}{na_nx^n + (n-1)a_{n-1}x^{n-1} + \cdots + a_1x} = 0$$

Thus $\ln x$ grows slower than any nonconstant

polynomial as $x\to\infty$.

39. Compare $n \log_2 n$ to $n^{3/2}$ as $n\to\infty$.

$$\lim_{n\to\infty} \frac{n \log_2 n}{n^{3/2}} = \lim_{n\to\infty} \frac{\log_2 n}{n^{1/2}}$$

$$= \lim_{n\to\infty} \frac{\frac{(\ln n)}{(\ln 2)}}{n^{1/2}}$$

$$= \lim_{n\to\infty} \frac{\frac{1}{n \ln 2}}{\frac{1}{2n^{1/2}}}$$

$$= \lim_{n\to\infty} \frac{2}{n^{1/2}(\ln 2)} = 0$$

Thus $n \log_2 n$ grows slower than $n^{3/2}$ as $n\to\infty$.

Compare $n \log_2 n$ to $n(\log_2 n)^2$

$$\lim_{n\to\infty} \frac{n \log_2 n}{n(\log_2 n)^2} = \lim_{n\to\infty} \frac{1}{\log_2 n} = 0$$

Thus $n \log_2 n$ grows slower than $n(\log_2 n)^2$ as $n\to\infty$. The algorithm of order of $n \log_2 n$ is likely the most efficient because of the three functions, it grows the most slowly as $n\to\infty$.

40. **(a)** It might take 1,000,000 searches if it is the last item in the search.

(b) $\log_2 1{,}000{,}000 \approx 19.9$; it might take 20 binary searches.

41. Since f and g grow at the same rate, there exists a nonzero number L such that $\lim_{x\to\infty} \frac{f(x)}{g(x)} = L$. Then for sufficiently large x, $\frac{f(x)}{g(x)} < L + 1 \le M$ for some integer M. Similarly, for sufficiently large x, $\frac{g(x)}{f(x)} < \frac{1}{L} + 1 \le N$ for some integer N.

42. **(a)** The limit will be the ratio of the leading coefficients of the polynomials since the polynomials must have the same degree.

(b) By the same reason as in (a), the limit will be the ratio of the leading coefficients of the polynomial.

43. **(a)** $\lim_{x\to\infty} \frac{x^5}{x^2} = \lim_{x\to\infty} x^3 = \infty$

x^5 grows faster than x^2.

(b) $\lim_{x\to\infty} \frac{5x^3}{2x^3} = \lim_{x\to\infty} \frac{5}{2} = \frac{5}{2}$

$5x^3$ and $2x^3$ have the same rate of growth.

(c) $m > n$ since $\lim_{x\to\infty} \frac{x^m}{x^n} = \lim_{x\to\infty} x^{m-n} = \infty$.

(d) $m = n$ since $\lim_{x\to\infty} \frac{x^m}{x^n} = \lim_{x\to\infty} x^{m-n}$ is nonzero and finite.

(e) Degree of $g >$ degree of f $(m > n)$ since $\lim_{x\to\infty} \frac{g(x)}{f(x)} = \infty$.

(f) Degree of $g =$ degree of f $(m = n)$ since $\lim_{x\to\infty} \frac{g(x)}{f(x)}$ is nonzero and finite.

44. (a) $f = o(g)$ as $x \to a$ if $\lim_{x \to a} \frac{f(x)}{g(x)} = 0$

Suppose f and g are both positive in some open interval containing a. Then $f = O(g)$ as $x \to a$ if there is a positive integer M for which $\frac{f(x)}{g(x)} \le M$ for x sufficiently close to a.

(b) From Section 5.5, we know that $|E_S| \le \frac{b-a}{180} h^4 M$ where M is a bound for the absolute value of $f^{(4)}$ on $[a, b]$. Thus, $\frac{|E_S|}{h^4} \le (b-a)\frac{M}{180} \le \text{int}\left[(b-a)\frac{M}{180}\right] + 1$ as $h \to 0$, so $|E_S| = O(h^4)$. Thus as $h \to 0$, $E_S \to 0$.

(c) From Section 5.6, we know that $|E_T| \le \frac{b-a}{12} h^2 M$ where M is a bound for the absolute value of f'' on $[a, b]$. Thus $\frac{|E_T|}{h^2} \le (b-a)\frac{M}{12} \le \text{int}\left[(b-a)\frac{M}{12}\right] + 1$ as $h \to 0$, so $|E_T| = O(h^2)$. Thus as $h \to 0$, $E_T \to 0$.

45. (a) $\lim_{x\to\infty} \frac{|f(x)|}{|g(x)|} = \lim_{x\to\infty} \frac{-f(x)}{-g(x)} = \lim_{x\to\infty} \frac{f(x)}{g(x)} = \infty$

Thus $|f|$ grows faster than $|g|$ as $x \to \infty$ by definition.

(b) $\lim_{x\to\infty} \frac{|f(x)|}{|g(x)|} = \lim_{x\to\infty} \frac{-f(x)}{-g(x)} = \lim_{x\to\infty} \frac{f(x)}{g(x)} = L$

Thus $|f|$ grows at the same rate as $|g|$ as $x \to \infty$ by definition.

46. (a) $\lim_{x\to\infty} \frac{f(-x)}{g(-x)} = \lim_{x\to-\infty} \frac{f(x)}{g(x)} = \infty$

Thus $f(-x)$ grows faster than $g(-x)$ by definition.

(b) $\lim_{x\to\infty} \frac{f(-x)}{g(-x)} = \lim_{x\to-\infty} \frac{f(x)}{g(x)} = L$

Thus $f(-x)$ grows at the same rate as $g(x)$ by definition.

■ Section 8.3 Improper Integrals
(pp. 433–444)

Exploration 1 Investigating $\int_0^1 \frac{dx}{x^p}$

1. Because $\frac{1}{x^p}$ has an infinite discontinuity at $x = 0$.

2. $\int_0^1 \frac{dx}{x} = \lim_{c\to0^+} \int_c^1 \frac{dx}{x} = \lim_{c\to0^+} \Big[\ln x\Big]_c^1 = \lim_{c\to0^+} (-\ln c) = \infty$

3. If $p > 1$, then

$$\begin{aligned}\int_0^1 \frac{dx}{x^p} &= \lim_{c\to0^+} \int_c^1 \frac{dx}{x^p}\\ &= \lim_{c\to0^+} \frac{x^{-p+1}}{-p+1}\Bigg]_c^1\\ &= \lim_{c\to0^+} \left(\frac{1-c^{-p+1}}{-p+1}\right) = \infty \text{ because } (-p+1) < 0.\end{aligned}$$

4. If $0 < p < 1$, then

$$\begin{aligned}\int_0^1 \frac{dx}{x^p} &= \lim_{c\to0^+} \int_c^1 \frac{dx}{x^p}\\ &= \lim_{c\to0^+} \frac{x^{-p+1}}{-p+1}\Bigg]_c^1\\ &= \lim_{c\to0^+} \left(\frac{1-c^{-p+1}}{-p+1}\right) = \frac{1}{1-p}\end{aligned}$$

Quick Review 8.3

1. $\int_0^3 \frac{dx}{x+3} = \Big[\ln|x+3|\Big]_0^3 = \ln 6 - \ln 3 = \ln 2$

2. $\int_{-1}^1 \frac{x\,dx}{x^2+1} = \left[\frac{1}{2}\ln|x^2+1|\right]_{-1}^1 = \frac{1}{2}\ln 2 - \frac{1}{2}\ln 2 = 0$

3.
$$\begin{aligned}\int \frac{dx}{x^2+4} &= \frac{1}{4}\int \frac{dx}{\left(\frac{x}{2}\right)^2 + 1}\\ &= \frac{1}{4}\left(2\tan^{-1}\frac{x}{2}\right) + C\\ &= \frac{1}{2}\tan^{-1}\frac{x}{2} + C\end{aligned}$$

4. $\int \frac{dx}{x^4} = \int x^{-4}\,dx = -\frac{1}{3}x^{-3} + C$

5. $9 - x^2 > 0$ for $-3 < x < 3$

The domain is $(-3, 3)$.

6. $x - 1 > 0$ for $x > 1$

The domain is $(1, \infty)$.

7. $-1 \le \cos x \le 1$, so $|\cos x| \le 1$.

$\left|\frac{\cos x}{x^2}\right| = \frac{|\cos x|}{|x^2|} \le \frac{1}{x^2}$

8. $x^2 - 1 \le x^2$ so $\sqrt{x^2-1} \le \sqrt{x^2} = x$ for $x > 1$

$\frac{1}{\sqrt{x^2-1}} \ge \frac{1}{x}$

9. $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{4e^x - 5}{3e^x + 7} = \lim_{x\to\infty} \frac{4e^x}{3e^x} = \lim_{x\to\infty} \frac{4}{3} = \frac{4}{3}$

Thus f and g grow at the same rate as $x \to \infty$.

10.
$$\begin{aligned}\lim_{x\to\infty} \frac{f(x)}{g(x)} &= \lim_{x\to\infty} \frac{\sqrt{2x-1}}{\sqrt{x+3}}\\ &= \lim_{x\to\infty} \sqrt{\frac{2x-1}{x+3}}\\ &= \lim_{x\to\infty} \sqrt{\frac{2 - \frac{1}{x}}{1 + \frac{3}{x}}} = \sqrt{2}\end{aligned}$$

Section 8.3 Exercises

1. (a) The integral is improper because of an infinite limit of integration.

(b)
$$\int_0^\infty \frac{dx}{x^2+1} = \lim_{b\to\infty} \int_0^b \frac{dx}{x^2+1}$$
$$= \lim_{b\to\infty} \Big[\tan^{-1} x\Big]_0^b$$
$$= \lim_{b\to\infty} (\tan^{-1} b - 0)$$
$$= \frac{\pi}{2}$$

The integral converges.

(c) $\frac{\pi}{2}$

2. (a) The integral is improper because the integrand has an infinite discontinuity at $x = 0$.

(b)
$$\int_0^1 \frac{dx}{\sqrt{x}} = \lim_{b\to 0^+} \int_b^1 \frac{dx}{\sqrt{x}}$$
$$= \lim_{b\to 0^+} \Big[2\sqrt{x}\Big]_b^1$$
$$= \lim_{b\to 0^+} (2 - 2\sqrt{b}) = 2$$

The integral converges.

(c) 2

3. (a) The integral involves improper integrals because the integrand has an infinite discontinuity at $x = 0$.

(b)
$$\int_{-8}^1 \frac{dx}{x^{1/3}} = \int_{-8}^0 \frac{dx}{x^{1/3}} + \int_0^1 \frac{dx}{x^{1/3}}$$
$$\int_{-8}^0 \frac{dx}{x^{1/3}} = \lim_{b\to 0^-} \int_{-8}^b \frac{dx}{x^{1/3}}$$
$$= \lim_{b\to 0^-} \left[\frac{3}{2}x^{2/3}\right]_{-8}^b$$
$$= \lim_{b\to 0^-} \left(\frac{3}{2}b^{2/3} - 6\right) = -6$$
$$\int_0^1 \frac{dx}{x^{1/3}} = \lim_{b\to 0^+} \int_b^1 \frac{dx}{x^{1/3}}$$
$$= \lim_{b\to 0^+} \left[\frac{3}{2}x^{2/3}\right]_b^1$$
$$= \lim_{b\to 0^+} \left(\frac{3}{2} - \frac{3}{2}b^{2/3}\right)$$
$$= \frac{3}{2}$$
$$\int_{-8}^1 \frac{dx}{x^{1/3}} = -6 + \frac{3}{2} = -\frac{9}{2}$$

The integral converges.

(c) $-\frac{9}{2}$

4. (a) The integral is improper because of two infinite limits of integration.

(b)
$$\int_{-\infty}^\infty \frac{2x\,dx}{(x^2+1)^2} = \int_{-\infty}^0 \frac{2x\,dx}{(x^2+1)^2} + \int_0^\infty \frac{2x\,dx}{(x^2+1)^2}$$
$$\int_{-\infty}^0 \frac{2x\,dx}{(x^2+1)^2} = \lim_{b\to-\infty} \int_b^0 \frac{2x\,dx}{(x^2+1)^2}$$
$$= \lim_{b\to-\infty} \Big[-(x^2+1)^{-1}\Big]_b^0$$
$$= \lim_{b\to-\infty} [-1 + (b^2+1)^{-1}] = -1$$
$$\int_0^\infty \frac{2x\,dx}{(x^2+1)^2} = \lim_{b\to\infty} \int_0^b \frac{2x\,dx}{(x^2+1)^2}$$
$$= \lim_{b\to\infty} \Big[(-x^2+1)^{-1}\Big]_0^b$$
$$= \lim_{b\to\infty} [-(b^2+1)^{-1} + 1] = 1$$
$$\int_{-\infty}^\infty \frac{2x\,dx}{(x^2+1)^2} = -1 + 1 = 0$$

The integral converges.

(c) 0

5. (a) The integral is improper because the integrand has an infinite discontinuity at 0.

(b)
$$\int_0^{\ln 2} x^{-2} e^{1/x}\,dx = \lim_{b\to 0^+} \int_b^{\ln 2} x^{-2} e^{1/x}\,dx$$
$$= \lim_{b\to 0^+} \Big[-e^{1/x}\Big]_b^{\ln 2}$$
$$= \lim_{b\to 0^+} [-e^{1/\ln 2} + e^{1/b}] = \infty$$

The integral diverges.

(c) No value

6. (a) The integral is improper because the integrand has an infinite discontinuity at $x = 0$.

(b)
$$\int_0^{\pi/2} \cot\theta\,d\theta = \lim_{b\to 0^+} \int_b^{\pi/2} \cot\theta\,d\theta$$
$$= \lim_{b\to 0^+} \int_b^{\pi/2} \frac{\cos\theta\,d\theta}{\sin\theta}$$
$$= \lim_{b\to 0^+} \Big[\ln|\sin\theta|\Big]_b^{\pi/2}$$
$$= \lim_{b\to 0^+} (0 - \ln|\sin b|) = \infty$$

The integral diverges.

(c) No value

7.
$$\int_1^\infty \frac{dx}{x^{1.001}} = \lim_{b\to\infty} \int_1^b \frac{dx}{x^{1.001}}$$
$$= \lim_{b\to\infty} \Big[-1000\,x^{-0.001}\Big]_1^b$$
$$= \lim_{b\to\infty} (-1000b^{-0.001} + 1000) = 1000$$

8. $\int_{-1}^{1} \frac{dx}{x^{2/3}} = \int_{-1}^{0} \frac{dx}{x^{2/3}} + \int_{0}^{1} \frac{dx}{x^{2/3}}$

$$\begin{aligned}\int_{-1}^{0} \frac{dx}{x^{2/3}} &= \lim_{b\to 0^-} \int_{-1}^{b} \frac{dx}{x^{2/3}}\\ &= \lim_{b\to 0^-} \Big[3x^{1/3}\Big]_{-1}^{b}\\ &= \lim_{b\to 0^-} (3b^{1/3} + 3) = 3\end{aligned}$$

$$\begin{aligned}\int_{b}^{1} \frac{dx}{x^{2/3}} &= \lim_{b\to 0^+} \int_{b}^{1} \frac{dx}{x^{2/3}}\\ &= \lim_{b\to 0^+} \Big[3x^{1/3}\Big]_{b}^{1}\\ &= \lim_{b\to 0^+} (3 - 3b^{1/3}) = 3\end{aligned}$$

$$\int_{-1}^{1} \frac{dx}{x^{2/3}} = 3 + 3 = 6$$

9. $$\begin{aligned}\int_{0}^{4} \frac{dr}{\sqrt{4-r}} &= \lim_{b\to 4^-} \int_{0}^{b} \frac{dr}{\sqrt{4-r}}\\ &= \lim_{b\to 4^-} \Big[-2\sqrt{4-r}\Big]_{0}^{b}\\ &= \lim_{b\to 4^-} (-2\sqrt{4-b} + 4) = 4\end{aligned}$$

10. $$\begin{aligned}\int_{0}^{1} \frac{dr}{r^{0.999}} &= \lim_{b\to 0^+} \int_{b}^{1} \frac{dr}{r^{0.999}}\\ &= \lim_{b\to 0^+} \Big[1000r^{0.001}\Big]_{b}^{1}\\ &= \lim_{b\to 0^+} (1000 - 1000b^{0.001}) = 1000\end{aligned}$$

11. $$\begin{aligned}\int_{0}^{1} \frac{dx}{\sqrt{1-x^2}} &= \lim_{b\to 1^-} \int_{0}^{b} \frac{dx}{\sqrt{1-x^2}}\\ &= \lim_{b\to 1^-} \Big[\sin^{-1} x\Big]_{0}^{b}\\ &= \lim_{b\to 1^-} (\sin^{-1} b - 0) = \frac{\pi}{2}\end{aligned}$$

12. $$\begin{aligned}\int_{-\infty}^{2} \frac{2\,dx}{x^2+4} &= \lim_{b\to -\infty} \int_{b}^{2} \frac{(1/2)\,dx}{(x/2)^2+1}\\ &= \lim_{b\to -\infty} \left[\tan^{-1} \frac{x}{2}\right]_{b}^{2}\\ &= \lim_{b\to -\infty} \left(\frac{\pi}{4} - \tan^{-1} \frac{b}{2}\right)\\ &= \frac{\pi}{4} + \frac{\pi}{2} = \frac{3\pi}{4}\end{aligned}$$

13. $$\begin{aligned}\int_{-\infty}^{-2} \frac{2\,dx}{x^2-1} &= \lim_{b\to -\infty} \int_{b}^{-2} \frac{[(x+1)-(x-1)]\,dx}{(x+1)(x-1)}\\ &= \lim_{b\to -\infty} \int_{b}^{-2} \left(\frac{1}{x-1} - \frac{1}{x+1}\right) dx\\ &= \lim_{b\to -\infty} \Big[\ln|x-1| - \ln|x+1|\Big]_{b}^{-2}\\ &= \lim_{b\to -\infty} \left[\ln\left|\frac{x-1}{x+1}\right|\right]_{b}^{-2}\\ &= \lim_{b\to -\infty} \left(\ln 3 - \ln\left|\frac{b-1}{b+1}\right|\right)\\ &= \ln 3 - \ln 1 = \ln 3\end{aligned}$$

14. $$\begin{aligned}\int_{2}^{\infty} \frac{3\,dt}{t^2-t} &= \lim_{b\to\infty} \int_{2}^{\infty} \frac{3[t-(t-1)]\,dt}{t(t-1)}\\ &= \lim_{b\to\infty} \int_{2}^{\infty} \left(\frac{3}{t-1} - \frac{3}{t}\right) dt\\ &= \lim_{b\to\infty} \Big[3\ln|t-1| - 3\ln|t|\Big]_{2}^{b}\\ &= \lim_{b\to\infty} \left[3\ln\left|\frac{t-1}{t}\right|\right]_{2}^{b}\\ &= \lim_{b\to\infty} \left(3\ln\left|\frac{b-1}{b}\right| - 3\ln\frac{1}{2}\right)\\ &= 3\ln 1 + 3\ln 2 = 3\ln 2\end{aligned}$$

15. $$\begin{aligned}\int_{0}^{1} \frac{\theta+1}{\sqrt{\theta^2+2\theta}} &= \lim_{b\to 0^+} \int_{b}^{1} \frac{1\,(2\theta+2)\,d\theta}{2\,\sqrt{\theta^2+2\theta}}\\ &= \lim_{b\to 0^+} \Big[\sqrt{\theta^2+2\theta}\Big]_{b}^{1}\\ &= \lim_{b\to 0^+} (\sqrt{3} - \sqrt{b^2+2b}) = \sqrt{3}\end{aligned}$$

16. $\int_{0}^{2} \frac{s+1}{\sqrt{4-s^2}}\,ds = \lim_{b\to 2^-} \int_{0}^{b} \left(\frac{s}{\sqrt{4-s^2}} + \frac{1}{\sqrt{4-s^2}}\right) ds$

$$\begin{aligned}&\lim_{b\to 2^-} \int_{0}^{b} \left(\frac{s}{\sqrt{4-s^2}} + \frac{1}{2\sqrt{1-(s/2)^2}}\right) ds\\ &= \lim_{b\to 2^-} \left[-\sqrt{4-s^2} + \sin^{-1} \frac{s}{2}\right]_{0}^{b}\\ &= \lim_{b\to 2^-} \left(-\sqrt{4-b^2} + \sin^{-1} \frac{b}{2} + 2\right)\\ &= \sin^{-1} 1 + 2 = \frac{\pi}{2} + 2\end{aligned}$$

17. First integrate $\int \frac{dx}{(1+x)\sqrt{x}}$ by letting

$u = \sqrt{x}$, so $du = \frac{1}{2\sqrt{x}}\,dx$.

$$\begin{aligned}\int \frac{dx}{(1+x)\sqrt{x}} &= \int \frac{2\,du}{1+u^2}\\ &= 2\tan^{-1} u + C\\ &= 2\tan^{-1}\sqrt{x} + C\end{aligned}$$

Now evaluate the improper integral. Note that the integrand is infinite at $x = 0$.

$$\int_0^\infty \frac{dx}{(1+x)\sqrt{x}} = \int_0^1 \frac{dx}{(1+x)\sqrt{x}} + \int_1^\infty \frac{dx}{(1+x)\sqrt{x}}$$

$$= \lim_{b\to 0^+} \int_b^1 \frac{dx}{(1+x)\sqrt{x}} + \lim_{c\to\infty} \int_1^c \frac{dx}{(1+x)\sqrt{x}}$$

$$= \lim_{b\to 0^+} \Big[2\tan^{-1}\sqrt{x}\Big]_b^1 + \lim_{c\to\infty}\Big[2\tan^{-1}\sqrt{x}\Big]_1^c$$

$$= \lim_{b\to 0^+} (2\tan^{-1} 1 - 2\tan^{-1}\sqrt{b}) +$$

$$\lim_{c\to\infty} (2\tan^{-1}\sqrt{c} - 2\tan^{-1} 1)$$

$$= \left(\frac{\pi}{2} - 0\right) + \left(\pi - \frac{\pi}{2}\right) = \pi$$

18. $$\int_1^\infty \frac{dx}{x\sqrt{x^2-1}} = \int_1^2 \frac{dx}{x\sqrt{x^2-1}} + \int_2^\infty \frac{dx}{x\sqrt{x^2-1}}$$

$$\begin{aligned}\int_1^2 \frac{dx}{x\sqrt{x^2-1}} &= \lim_{b\to 1^+} \int_b^2 \frac{dx}{x\sqrt{x^2-1}}\\ &= \lim_{b\to 1^+} \Big[\sec^{-1} x\Big]_b^2\\ &= \lim_{b\to 1^+} (\sec^{-1} 2 - \sec^{-1} b)\\ &= \sec^{-1} 2 - \sec^{-1} 1 = \frac{\pi}{3}\end{aligned}$$

$$\begin{aligned}\int_2^\infty \frac{dx}{x\sqrt{x^2-1}} &= \lim_{b\to\infty} \int_2^b \frac{dx}{x\sqrt{x^2-1}}\\ &= \lim_{b\to\infty} \Big[\sec^{-1} x\Big]_2^b\\ &= \lim_{b\to\infty} (\sec^{-1} b - \sec^{-1} 2)\\ &= \frac{\pi}{2} - \frac{\pi}{3} = \frac{\pi}{6}\end{aligned}$$

$$\int_1^\infty \frac{dx}{x\sqrt{x^2-1}} = \frac{\pi}{3} + \frac{\pi}{6} = \frac{\pi}{2}$$

19. $$\begin{aligned}\int_1^2 \frac{ds}{s\sqrt{s^2-1}} &= \lim_{b\to 1^+} \int_b^2 \frac{ds}{s\sqrt{s^2-1}}\\ &= \lim_{b\to 1^+} \Big[\sec^{-1} s\Big]_b^2\\ &= \lim_{b\to 1^+} (\sec^{-1} 2 - \sec^{-1} b)\\ &= \sec^{-1} 2 - \sec^{-1} 1 = \frac{\pi}{3}\end{aligned}$$

20. $$\begin{aligned}\int_{-1}^\infty \frac{d\theta}{\theta^2 + 5\theta + 6} &= \lim_{b\to\infty} \int_{-1}^b \frac{(\theta+3) - (\theta+2)}{(\theta+3)(\theta+2)}\,d\theta\\ &= \lim_{b\to\infty} \int_{-1}^b \left(\frac{1}{\theta+2} - \frac{1}{\theta+3}\right) d\theta\\ &= \lim_{b\to\infty} \Big[\ln|\theta+2| - \ln|\theta+3|\Big]_{-1}^b\\ &= \lim_{b\to\infty} \left[\ln\left|\frac{\theta+2}{\theta+3}\right|\right]_{-1}^{b}\\ &= \lim_{b\to\infty} \left(\ln\left|\frac{b+2}{b+3}\right| - \ln\frac{1}{2}\right) = \ln 2\end{aligned}$$

21. Integrate $\int \frac{16\tan^{-1} x}{1+x^2}\,dx$ by letting $u = \tan^{-1} x$, so

$du = \frac{dx}{1+x^2}$.

$$\begin{aligned}\int \frac{16\tan^{-1} x}{1+x^2}\,dx &= \int 16u\,du = 8u^2 + C\\ &= 8(\tan^{-1} x)^2 + C\end{aligned}$$

$$\begin{aligned}\int_0^\infty \frac{16\tan^{-1} x}{1+x^2}\,dx &= \lim_{b\to\infty} \int_0^b \frac{16\tan^{-1} x}{1+x^2}\,dx\\ &= \lim_{b\to\infty} \Big[8\,(\tan^{-1} x)^2\Big]_0^b\\ &= \lim_{b\to\infty} \Big[8(\tan^{-1} b)^2 - 0\Big]\\ &= 8\left(\frac{\pi}{2}\right)^2 = 2\pi^2\end{aligned}$$

22. $$\int_{-1}^4 \frac{dx}{\sqrt{|x|}} = \int_{-1}^0 \frac{dx}{\sqrt{-x}} + \int_0^4 \frac{dx}{\sqrt{x}}$$

$$\begin{aligned}\int_{-1}^0 \frac{dx}{\sqrt{-x}} &= \lim_{b\to 0^-} \int_{-1}^b \frac{dx}{\sqrt{-x}}\\ &= \lim_{b\to 0^-} \Big[-2\sqrt{-x}\Big]_{-1}^b\\ &= \lim_{b\to 0^-} (-2\sqrt{-b} + 2) = 2\end{aligned}$$

$$\begin{aligned}\int_0^4 \frac{dx}{\sqrt{x}} &= \lim_{b\to 0^+} \int_b^4 \frac{dx}{\sqrt{x}}\\ &= \lim_{b\to 0^+} \Big[2\sqrt{x}\Big]_b^4\\ &= \lim_{b\to 0^+} \left(4 - 2\sqrt{b}\right) = 4\end{aligned}$$

$$\int_{-1}^4 \frac{dx}{\sqrt{|x|}} = 2 + 4 = 6$$

23. Integrate $\int \theta e^{\theta}\,d\theta$ by parts.

$u = \theta \qquad dv = e^{\theta}\,d\theta$

$du = d\theta \qquad v = e^{\theta}$

$$\int \theta e^{\theta}\,d\theta = \theta e^{\theta} - \int e^{\theta}\,d\theta = \theta e^{\theta} - e^{\theta} + C$$

$$\begin{aligned}\int_{-\infty}^{0} \theta e^{\theta}\,d\theta &= \lim_{b\to-\infty} \int_{b}^{0} \theta e^{\theta}\,d\theta \\ &= \lim_{b\to-\infty} \Big[\theta e^{\theta} - e^{\theta}\Big]_{b}^{0} \\ &= \lim_{b\to-\infty} (-1 - be^{b} + e^{b}) = -1\end{aligned}$$

$\left(\text{Note that } \lim_{b\to-\infty} be^{b} = \lim_{c\to\infty} -ce^{-c} = \lim_{c\to\infty} -\frac{c}{e^{c}} = \lim_{c\to\infty} -\frac{1}{e^{c}} = 0 \text{ and } \lim_{b\to-\infty} e^{b} = \lim_{c\to\infty} e^{-c} = 0.\right)$

24. Integrate $\int 2e^{-\theta}\sin\theta\,d\theta$ by parts.

$u = 2\sin\theta \qquad dv = e^{-\theta}\,d\theta$

$du = 2\cos\theta\,d\theta \qquad v = -e^{-\theta}$

$$\int 2e^{-\theta}\sin\theta\,d\theta = -2e^{-\theta}\sin\theta + \int 2e^{-\theta}\cos\theta\,d\theta$$

Integrate $\int 2e^{-\theta}\cos\theta\,d\theta$ by parts.

$u = 2\cos\theta \qquad dv = e^{-\theta}\,d\theta$

$du = -2\sin\theta\,d\theta \qquad v = -e^{-\theta}$

$$\int 2e^{-\theta}\cos\theta\,d\theta = -2e^{-\theta}\cos\theta - \int 2e^{-\theta}\sin\theta\,d\theta$$

Thus,

$$\begin{aligned}&\int 2e^{-\theta}\sin\theta\,d\theta \\ &\qquad = -2e^{-\theta}\sin\theta - 2e^{-\theta}\cos\theta - \int 2e^{-\theta}\sin\theta\,d\theta\end{aligned}$$

$$2\int 2e^{-\theta}\sin\theta\,d\theta = -2e^{-\theta}\sin\theta - 2e^{-\theta}\cos\theta + C_1$$

$$\int 2e^{-\theta}\sin\theta\,d\theta = -e^{-\theta}\sin\theta - e^{-\theta}\cos\theta + C$$

$$\begin{aligned}\int_{0}^{\infty} 2e^{-\theta}\sin\theta\,d\theta &= \lim_{b\to\infty}\int_{0}^{b} 2e^{-\theta}\sin\theta\,d\theta \\ &= \lim_{b\to\infty}\Big[-e^{-\theta}\sin\theta - e^{-\theta}\cos\theta\Big]_{0}^{b} \\ &= \lim_{b\to\infty} (-e^{-b}\sin b - e^{-b}\cos b + 1) = 1\end{aligned}$$

25. $\int_{-\infty}^{\infty} e^{-|x|}\,dx = \int_{-\infty}^{0} e^{x}\,dx + \int_{0}^{\infty} e^{-x}\,dx$

$$\int_{-\infty}^{0} e^{x}\,dx = \lim_{b\to-\infty}\int_{b}^{0} e^{x}\,dx = \lim_{b\to-\infty}\Big[e^{x}\Big]_{b}^{0} = \lim_{b\to-\infty}(1 - e^{b}) = 1$$

$$\begin{aligned}\int_{0}^{\infty} e^{-x}\,dx &= \lim_{b\to\infty}\int_{0}^{b} e^{-x}\,dx = \lim_{b\to\infty}\Big[-e^{-x}\Big]_{0}^{b} \\ &= \lim_{b\to\infty}(-e^{-b} + 1) = 1\end{aligned}$$

$$\int_{-\infty}^{\infty} e^{-|x|}\,dx = 1 + 1 = 2$$

26. Integrate $\int x\ln x\,dx$ by parts.

$u = \ln x \qquad dv = x\,dx$

$du = \frac{1}{x}\,dx \qquad v = \frac{1}{2}x^{2}$

$$\int x\ln x\,dx = \frac{1}{2}x^{2}\ln x - \int \frac{1}{2}x\,dx = \frac{1}{2}x^{2}\ln x - \frac{1}{4}x^{2} + C$$

$$\begin{aligned}\int_{0}^{1} x\ln x &= \lim_{b\to0^{+}}\int_{b}^{1} x\ln x\,dx \\ &= \lim_{b\to0^{+}}\left[\frac{1}{2}x^{2}\ln x - \frac{1}{4}x^{2}\right]_{b}^{1} \\ &= \lim_{b\to0^{+}}\left(-\frac{1}{4} - \frac{1}{2}b^{2}\ln b + \frac{1}{4}b^{2}\right) \\ &= -\frac{1}{4}\end{aligned}$$

$\left(\text{Note that } \lim_{b\to0^{+}} b^{2}\ln b = \lim_{b\to0^{+}} \frac{\ln b}{1/b^{2}} = \lim_{b\to0^{+}} \frac{1/b}{-2/b^{3}} = \lim_{b\to0^{+}} -\frac{b^{2}}{2} = 0.\right)$

27. $$\begin{aligned}\int_{0}^{\pi/2} \tan\theta\,d\theta &= \lim_{b\to\pi/2}\int_{0}^{b} \frac{\sin\theta}{\cos\theta}\,d\theta \\ &= \lim_{b\to\pi/2}\Big[-\ln|\cos\theta|\Big]_{0}^{b} \\ &= \lim_{b\to\pi/2}\left[-\ln|\cos b| + 0\right] = \infty\end{aligned}$$

The integral diverges.

28. On $[0, \pi]$, $0 \le \dfrac{\sin\theta}{\sqrt{\pi-\theta}} \le \dfrac{1}{\sqrt{\pi-\theta}}$, so

$$\int_0^{\pi} \frac{\sin\theta}{\sqrt{\pi-\theta}}\,d\theta \le \int_0^{\pi} \frac{1}{\sqrt{\pi-\theta}}\,d\theta$$

$$\begin{aligned}\int_0^{\pi} \frac{1}{\sqrt{\pi-\theta}}\,d\theta &= \lim_{b\to\pi^-} \int_0^b \frac{1}{\sqrt{\pi-\theta}}\,d\theta\\ &= \lim_{b\to\pi^-} \left[-2\sqrt{\pi-\theta}\right]_0^b\\ &= \lim_{b\to\pi^-} \left(-2\sqrt{\pi-b} + 2\sqrt{\pi}\right)\\ &= -2\sqrt{0} + 2\sqrt{\pi}\\ &= 2\sqrt{\pi}\end{aligned}$$

Since this integral converges, the given integral converges.

29. $\displaystyle\int_{-\infty}^{\infty} 2xe^{-x^2}\,dx = \int_0^{\infty} 2xe^{-x^2}\,dx + \int_{-\infty}^{0} 2xe^{-x^2}\,dx$

$$\begin{aligned}\int_0^{\infty} 2xe^{-x^2}\,dx &= \lim_{b\to\infty} \int_0^b 2xe^{-x^2}\,dx\\ &= \lim_{b\to\infty} \left[-e^{-x^2}\right]_0^b\\ &= \lim_{b\to\infty} \left[-e^{-b^2} + 1\right] = 1\end{aligned}$$

$$\begin{aligned}\int_{-\infty}^{0} 2xe^{-x^2}\,dx &= \lim_{b\to-\infty} \int_b^0 2xe^{-x^2}\,dx\\ &= \lim_{b\to-\infty} \left[-e^{-x^2}\right]_b^0\\ &= \lim_{b\to-\infty} \left[-1 + e^{-b^2}\right] = -1\end{aligned}$$

The integral converges.

30.
$$\begin{aligned}\int_0^4 \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,dx &= \lim_{b\to0^+} \int_b^4 \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,dx\\ &= \lim_{b\to0^+} \left[-2e^{-\sqrt{x}}\right]_b^4\\ &= \lim_{b\to0^+} \left[-2e^{-2} + 2e^{-\sqrt{b}}\right]\\ &= -2e^{-2} + 2\end{aligned}$$

The integral converges.

31. $0 \le \dfrac{1}{\sqrt{t} + \sin t} \le \dfrac{1}{\sqrt{t}}$ on $(0, \pi]$ since $\sin t \ge 0$ on $[0, \pi]$.

$$\begin{aligned}\int_0^{\pi} \frac{dt}{\sqrt{t}} &= \lim_{b\to0^+} \int_b^{\pi} \frac{dt}{\sqrt{t}}\\ &= \lim_{b\to0^+} \left[2\sqrt{t}\right]_b^{\pi}\\ &= \lim_{b\to0^+} \left[2\sqrt{\pi} - 2\sqrt{b}\right]\\ &= 2\sqrt{\pi}\end{aligned}$$

Since this integral converges, the given integral converges.

32. $0 \le \dfrac{1}{\sqrt{x}} \le \dfrac{1}{\sqrt{x} - 1}$ on $[4, \infty)$

$$\int_4^{\infty} \frac{dx}{\sqrt{x}} = \lim_{b\to\infty} \int_4^b \frac{dx}{\sqrt{x}} = \lim_{b\to\infty} \left[2\sqrt{x}\right]_4^b = \lim_{b\to\infty} \left[2\sqrt{b} - 4\right] = \infty$$

Since this integral diverges, the given integral diverges.

33. $0 \le \dfrac{1}{x^3 + 1} \le \dfrac{1}{x^3}$ on $[1, \infty)$

$$\begin{aligned}\int_1^{\infty} \frac{dx}{x^3} &= \lim_{b\to\infty} \int_1^b \frac{dx}{x^3}\\ &= \lim_{b\to\infty} \left[-\frac{1}{2}x^{-2}\right]_1^b\\ &= \lim_{b\to\infty} \left[-\frac{1}{2}b^{-2} + \frac{1}{2}\right] = \frac{1}{2}\end{aligned}$$

Since this integral converges, the given integral converges.

34. $\displaystyle\int_0^2 \frac{dx}{1 - x^2} = \int_0^1 \frac{dx}{1 - x^2} + \int_1^2 \frac{dx}{1 - x^2}$

$$\begin{aligned}\int_0^1 \frac{dx}{1 - x^2} &= \lim_{b\to1^-} \int_0^b \frac{1/2[(1 - x) + (1 + x)]}{(1 - x)(1 + x)}\,dx\\ &= \lim_{b\to1^-} \int_0^b \left[\frac{1}{2(1 + x)} + \frac{1}{2(1 - x)}\right] dx\\ &= \lim_{b\to1^-} \left[\frac{1}{2}\ln|1 + x| - \frac{1}{2}\ln|1 - x|\right]_0^b\\ &= \lim_{b\to1^-} \left[\frac{1}{2}\ln\left|\frac{1 + x}{1 - x}\right|\right]_0^b\\ &= \lim_{b\to1^-} \left[\frac{1}{2}\ln\left|\frac{1 + b}{1 - b}\right| - 0\right] = \infty\end{aligned}$$

Since this integral diverges, the given integral diverges.

35. $\displaystyle\int_0^2 \frac{dx}{1 - x} = \int_0^1 \frac{dx}{1 - x} + \int_1^2 \frac{dx}{1 - x}$

$$\begin{aligned}\int_0^1 \frac{dx}{1 - x} &= \lim_{b\to1^-} \int_0^b \frac{dx}{1 - x}\\ &= \lim_{b\to1^-} \left[-\ln|1 - x|\right]_0^b\\ &= \lim_{b\to1^-} \left(-\ln|1 - b| + 0\right) = \infty\end{aligned}$$

Since this integral diverges, the given integral diverges.

36. $\int_{-1}^{1} \ln|x|\,dx = 2\int_{0}^{1} \ln x\,dx$ by symmetry of $\ln|x|$ about the y-axis. Integrate $\int \ln x\,dx$ by parts.

$u = \ln x \qquad dv = dx$

$du = \frac{1}{x}\,dx \qquad v = x$

$$\int \ln x\,dx = x\ln x - \int dx = x\ln x - x + C$$

$$\begin{aligned} 2\int_0^1 \ln x\,dx &= 2\lim_{b\to 0^+}\int_b^1 \ln x\,dx \\ &= 2\lim_{b\to 0^+}\Big[x\ln x - x\Big]_b^1 \\ &= 2\lim_{b\to 0^+}[-1 - b\ln b + b] = -2 \end{aligned}$$

$\left(\text{Note that } \lim_{b\to 0^+} b\ln b = \lim_{b\to 0^+}\frac{\ln b}{1/b} = \lim_{b\to 0^+}\frac{1/b}{-1/b^2} = \lim_{b\to 0^+} -b = 0.\right)$

The integral converges.

37. $0 \le \frac{1}{1+e^{\theta}} \le \frac{1}{e^{\theta}}$ on $[1, \infty)$

$$\begin{aligned} \int_1^{\infty}\frac{1}{e^{\theta}}\,d\theta &= \lim_{b\to\infty}\int_1^b e^{-\theta}\,d\theta \\ &= \lim_{b\to\infty}\Big[-e^{-\theta}\Big]_1^b \\ &= \lim_{b\to\infty}[-e^{-b} + e^{-1}] \\ &= \frac{1}{e} \end{aligned}$$

Since this integral converges, the given integral converges.

38. $0 \le \frac{1}{x} \le \frac{1}{\sqrt{x^2-1}}$ on $[2, \infty)$

$$\int_2^{\infty}\frac{dx}{x} = \lim_{b\to\infty}\int_2^b \frac{dx}{x} = \lim_{b\to\infty}\Big[\ln x\Big]_2^b = \lim_{b\to\infty}(\ln b - \ln 2) = \infty$$

Since this integral diverges, the given integral diverges.

39. Let $f(x) = \frac{\sqrt{x+1}}{x^2}$ and $g(x) = \frac{1}{x^{3/2}}$. Both are continuous on $[1, \infty)$.

$$\lim_{x\to\infty}\frac{f(x)}{g(x)} = \lim_{x\to\infty}\frac{\sqrt{x+1}}{\sqrt{x}} = \lim_{x\to\infty}\sqrt{1+\frac{1}{x}} = 1$$

$$\begin{aligned} \int_1^{\infty}\frac{1}{x^{3/2}}\,dx &= \lim_{b\to\infty}\int_1^b x^{-3/2}\,dx \\ &= \lim_{b\to\infty}\Big[-2x^{-1/2}\Big]_1^b \\ &= \lim_{b\to\infty}(-2b^{-1/2} + 2) = 2 \end{aligned}$$

Since this integral converges, the given integral converges.

40. $\int_0^{\infty}\frac{dx}{\sqrt{x}} = \int_0^1\frac{dx}{\sqrt{x}} + \int_1^{\infty}\frac{dx}{\sqrt{x}}$

$$\begin{aligned} \int_1^{\infty}\frac{dx}{\sqrt{x}} &= \lim_{b\to\infty}\int_1^b\frac{dx}{\sqrt{x}} \\ &= \lim_{b\to\infty}\Big[2\sqrt{x}\Big]_1^b \\ &= \lim_{b\to\infty}(2\sqrt{b} - 2) = \infty \end{aligned}$$

Since this integral diverges, the given integral diverges.

41. $0 \le \frac{1}{x} \le \frac{2+\cos x}{x}$ on $[\pi, \infty)$

$$\int_{\pi}^{\infty}\frac{dx}{x} = \lim_{b\to\infty}\int_{\pi}^b\frac{dx}{x} = \lim_{b\to\infty}\Big[\ln x\Big]_{\pi}^b = \lim_{b\to\infty}(\ln b - \ln \pi) = \infty$$

Since this integral diverges, the given integral diverges.

42. $0 \le \frac{1+\sin x}{x^2} \le \frac{2}{x^2}$ on $[\pi, \infty)$

$$\begin{aligned} \int_{\pi}^{\infty}\frac{2\,dx}{x^2} &= \lim_{b\to\infty}\int_{\pi}^b 2x^{-2}\,dx \\ &= \lim_{b\to\infty}\Big[-2x^{-1}\Big]_{\pi}^b \\ &= \lim_{b\to\infty}\left(-2b^{-1} + \frac{2}{\pi}\right) = \frac{2}{\pi} \end{aligned}$$

Since this integral converges, the given integral converges.

43. First rewrite $\frac{1}{e^x + e^{-x}}$.

$$\frac{1}{e^x + e^{-x}} = \frac{1}{e^{-x}(e^{2x}+1)} = \frac{e^x}{1+(e^x)^2}$$

Integrate $\int\frac{e^x\,dx}{1+(e^x)^2}$ by letting $u = e^x$ so $du = e^x\,dx$.

$$\begin{aligned} \int\frac{dx}{e^x+e^{-x}} &= \int\frac{e^x\,dx}{1+(e^x)^2} \\ &= \int\frac{du}{1+u^2} \\ &= \tan^{-1}u + C \\ &= \tan^{-1}e^x + C \end{aligned}$$

$$\int_{-\infty}^{\infty}\frac{dx}{e^x+e^{-x}} = \int_{-\infty}^{0}\frac{dx}{e^x+e^{-x}} + \int_0^{\infty}\frac{dx}{e^x+e^{-x}}$$

$$\begin{aligned} \int_{-\infty}^{0}\frac{dx}{e^x+e^{-x}} &= \lim_{b\to-\infty}\int_b^0\frac{dx}{e^x+e^{-x}} \\ &= \lim_{b\to-\infty}\Big[\tan^{-1}e^x\Big]_b^0 \\ &= \lim_{b\to-\infty}[\tan^{-1}1 - \tan^{-1}e^b] \\ &= \frac{\pi}{4} - 0 = \frac{\pi}{4} \end{aligned}$$

$$\begin{aligned} \int_0^{\infty}\frac{dx}{e^x+e^{-x}} &= \lim_{b\to\infty}\int_0^b\frac{dx}{e^x+e^{-x}} \\ &= \lim_{b\to\infty}\Big[\tan^{-1}e^x\Big]_0^b \\ &= \lim_{b\to\infty}[\tan^{-1}e^b - \tan^{-1}1] \\ &= \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4} \end{aligned}$$

Thus, the given integral converges.

44. $\int_{-\infty}^{\infty} \frac{dx}{\sqrt{x^4+1}} = 2\int_{0}^{\infty} \frac{dx}{\sqrt{x^4+1}}$ by symmetry about the y-axis

$$\int_{0}^{\infty} \frac{dx}{\sqrt{x^4+1}} = \int_{0}^{1} \frac{dx}{\sqrt{x^4+1}} + \int_{1}^{\infty} \frac{dx}{\sqrt{x^4+1}}$$

$\int_{0}^{1} \frac{dx}{\sqrt{x^4+1}}$ exists because $\frac{1}{\sqrt{x^4+1}}$ exists on [0, 1].

$0 \le \frac{1}{\sqrt{x^4+1}} \le \frac{1}{x^2}$ on $[1, \infty)$.

$$\begin{aligned}\int_{1}^{\infty} \frac{1}{x^2}\,dx &= \lim_{b\to\infty} \int_{1}^{b} x^{-2}\,dx \\ &= \lim_{b\to\infty} \Big[-x^{-1}\Big]_{1}^{b} \\ &= \lim_{b\to\infty} \left[-\frac{1}{b} + 1\right] = 1\end{aligned}$$

Since this integral converges, the given integral converges.

45. Integrate $\int \frac{dy}{(1+y^2)(1+\tan^{-1} y)}$ by letting $u = \tan^{-1} y$ so

$du = \frac{dy}{1+y^2}$

$$\begin{aligned}\int \frac{dy}{(1+y^2)(1+\tan^{-1} y)} &= \int \frac{du}{1+u} \\ &= \ln |1+u| + C \\ &= \ln |1+\tan^{-1} y| + C\end{aligned}$$

$$\begin{aligned}\int_{0}^{\infty} \frac{dy}{(1+y^2)(1+\tan^{-1} y)} &= \lim_{b\to\infty} \int_{0}^{b} \frac{dy}{(1+y^2)(1+\tan^{-1} y)} \\ &= \lim_{b\to\infty} \Big[\ln |1+\tan^{-1} y|\Big]_{0}^{b} \\ &= \lim_{b\to\infty} (\ln |1+\tan^{-1} b| - 0) \\ &= \ln\left(1+\frac{\pi}{2}\right)\end{aligned}$$

The integral converges.

46. $\int_{-\infty}^{\infty} \frac{e^{-y}\,dy}{y^2+1} = \int_{-\infty}^{0} \frac{e^{-y}\,dy}{y^2+1} + \int_{0}^{\infty} \frac{e^{-y}\,dy}{y^2+1}$

$\int_{-\infty}^{0} \frac{e^{-y}\,dy}{y^2+1}$ diverges since

$$\lim_{y\to-\infty} \frac{e^{-y}}{y^2+1} = \lim_{y\to\infty} \frac{e^{y}}{y^2+1} = \lim_{y\to\infty} \frac{e^{y}}{2y} = \lim_{y\to\infty} \frac{e^{y}}{2} = \infty$$

Thus the given integral diverges.

47. For $x \ge 0$, $y \ge 0$ on $[1, \infty)$.

$$\text{Area} = \int_{1}^{\infty} \frac{\ln x}{x^2}\,dx = \lim_{b\to\infty} \int_{1}^{b} \frac{\ln x}{x^2}\,dx$$

Integrate $\int \frac{\ln x}{x^2}\,dx$ by parts.

$u = \ln x \qquad dv = \frac{dx}{x^2}$

$du = \frac{1}{x}\,dx \qquad v = -\frac{1}{x}$

$$\int \frac{\ln x}{x^2} = -\frac{\ln x}{x} + \int \frac{dx}{x^2} = -\frac{\ln x}{x} - \frac{1}{x} + C$$

$$\text{Area} = \lim_{b\to\infty} \left[-\frac{\ln x}{x} - \frac{1}{x}\right]_{1}^{b} = \lim_{b\to\infty} \left[-\frac{\ln b}{b} - \frac{1}{b} + 1\right] = 1$$

$\left(\text{Note that } \lim_{b\to\infty} \frac{\ln b}{b} = \lim_{b\to\infty} \frac{1/b}{1} = 0.\right)$

48. For $x \ge 0$, $y \ge 0$ on $[1, \infty)$.

$$\text{Area} = \int_{1}^{\infty} \frac{\ln x}{x}\,dx = \lim_{b\to\infty} \int_{1}^{b} \frac{\ln x}{x}\,dx$$

Integrate $\int \frac{\ln x}{x}\,dx$ by letting $u = \ln x$ so $du = \frac{dx}{x}$.

$$\int \frac{\ln x}{x}\,dx = \int u\,du = \frac{1}{2}u^2 + C = \frac{1}{2}(\ln x)^2 + C$$

$$\text{Area} = \lim_{b\to\infty} \left[\frac{1}{2}(\ln x)^2\right]_{1}^{b} = \lim_{b\to\infty} \frac{1}{2}(\ln b)^2 = \infty$$

49. **(a)** The integral in Example 1 gives the area of region R.

$$\text{Area} = \int_{1}^{\infty} \frac{dx}{x}$$

(b) Refer to Exploration 2 of Section 7.3.

$y' = -\frac{1}{x^2}$

The surface area of the solid is given by the following integral.

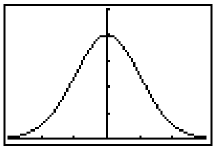
$$\begin{aligned}\int_{1}^{\infty} 2\pi\left(\frac{1}{x}\right)\sqrt{1+\left(-\frac{1}{x^2}\right)^2}\,dx &= 2\pi \int_{1}^{\infty} \frac{1}{x}\sqrt{\frac{x^4+1}{x^4}}\,dx \\ &= 2\pi \int_{1}^{\infty} \frac{\sqrt{x^4+1}}{x^3}\,dx\end{aligned}$$

Since $0 \le \frac{1}{x} \le \frac{\sqrt{x^4+1}}{x^3}$ on $[1, \infty)$, the direct comparison test shows that the integral for the surface area diverges. The surface area is ∞.

(c) $$\begin{aligned}\text{Volume} = \int_{1}^{\infty} \pi\left(\frac{1}{x}\right)^2 dx &= \pi\int_{1}^{\infty} \frac{1}{x^2}\,dx \\ &= \pi \lim_{b\to\infty} \int_{1}^{b} \frac{1}{x^2}\,dx \\ &= \pi \lim_{b\to\infty} \left[-\frac{1}{x}\right]_{1}^{b} \\ &= \pi \lim_{b\to\infty} \left(-\frac{1}{b} + 1\right) = \pi\end{aligned}$$

(d) Gabriel's horn has finite volume so it could only hold a finite amount of paint, but it has infinite surface area so it would require an infinite amount of paint to cover itself.

50. **(a)** $f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$



[−3, 3] by [0, 0.5]

f is increasing on $(-\infty, 0]$. f is decreasing on $[0, \infty)$.

f has a local maximum at $(0, f(0)) = \left(0, \frac{1}{\sqrt{2\pi}}\right)$

(b) $\text{NINT}\left(\frac{1}{\sqrt{2\pi}} e^{-x^2/2}, x, -1, 1\right) \approx 0.683$

$\text{NINT}\left(\frac{1}{\sqrt{2\pi}} e^{-x^2/2}, x, -2, 2\right) \approx 0.954$

$\text{NINT}\left(\frac{1}{\sqrt{2\pi}} e^{-x^2/2}, x, -3, 3\right) \approx 0.997$

(c) Part (b) suggests that as b increases, the integral approaches 1. We can make $\int_{-b}^{b} f(x)\,dx$ as close to 1 as we want by choosing $b > 1$ large enough. Also, we can make $\int_{b}^{\infty} f(x)\,dx$ and $\int_{-\infty}^{-b} f(x)\,dx$ as small as we want by choosing b large enough. This is because $0 < f(x) < e^{-x/2}$ for $x > 1$. (Likewise, $0 < f(x) < e^{x/2}$ for $x < -1$) Thus,

$$\int_{b}^{\infty} f(x)\,dx < \int_{b}^{\infty} e^{-x/2}\,dx$$

$$\begin{aligned}\int_{b}^{\infty} e^{-x/2}\,dx &= \lim_{c\to\infty} \int_{b}^{c} e^{-x/2}\,dx \\ &= \lim_{c\to\infty} \left[-2e^{-x/2}\right]_{b}^{c} \\ &= \lim_{c\to\infty} [-2e^{-c/2} + 2e^{-b/2}] \\ &= 2e^{-b/2}\end{aligned}$$

As $b\to\infty$, $2e^{-b/2}\to 0$, so for large enough b, $\int_{b}^{\infty} f(x)\,dx$ is as small as we want. Likewise, for large enough b, $\int_{-\infty}^{-b} f(x)\,dx$ is as small as we want.

51. **(a)** For $x \geq 6$, $x^2 \geq 6x$, so $e^{-x^2} \leq e^{-6x}$

$$\begin{aligned}\int_{6}^{\infty} e^{-x^2}\,dx \leq \int_{6}^{\infty} e^{-6x}\,dx &= \lim_{b\to\infty} \int_{6}^{b} e^{-6x}\,dx \\ &= \lim_{b\to\infty} \left[-\frac{1}{6}e^{-6x}\right]_{6}^{b} \\ &= \lim_{b\to\infty} \left(-\frac{1}{6}e^{-6b} + \frac{1}{6}e^{-36}\right) \\ &= \frac{1}{6}e^{-36} < 4 \times 10^{-17}\end{aligned}$$

(b) $$\begin{aligned}\int_{1}^{\infty} e^{-x^2}\,dx &= \int_{1}^{6} e^{-x^2}\,dx + \int_{6}^{\infty} e^{-x^2}\,dx \\ &\leq \int_{1}^{6} e^{-x^2}\,dx + 4 \times 10^{-17}\end{aligned}$$

Thus, from part (a) we have shown that the error is bounded by 4×10^{-17}.

(c) $\int_{1}^{\infty} e^{-x^2}\,dx \approx \text{NINT}(e^{-x^2}, x, 1, 6) \approx 0.1394027926$

(This agrees with Figure 8.16.)

(d) $$\begin{aligned}\int_{0}^{\infty} e^{-x^2}\,dx &= \int_{0}^{3} e^{-x^2}\,dx + \int_{3}^{\infty} e^{-x^2}\,dx \\ &\leq \int_{0}^{3} e^{-x^2}\,dx + \int_{3}^{\infty} e^{-3x}\,dx\end{aligned}$$

since $x^2 \geq 3x$ for $x > 3$.

$$\begin{aligned}\int_{3}^{\infty} e^{-3x}\,dx &= \lim_{b\to\infty} \int_{3}^{b} e^{-3x}\,dx \\ &= \lim_{b\to\infty} \left[-\frac{1}{3}e^{-3x}\right]_{3}^{b} \\ &= \lim_{b\to\infty} \left(-\frac{1}{3}e^{-3b} + \frac{1}{3}e^{-9}\right) \\ &= \frac{1}{3}e^{-9} \approx 0.000041 < 0.000042\end{aligned}$$

52. **(a)** Since f is even, $f(-x) = f(x)$. Let $u = -x$, $du = -dx$.

$$\begin{aligned}\int_{-\infty}^{\infty} f(x)\,dx &= \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{\infty} f(x)\,dx \\ &= \int_{\infty}^{0} f(-u)(-1)\,du + \int_{0}^{\infty} f(x)\,dx \\ &= \int_{0}^{\infty} f(u)\,du + \int_{0}^{\infty} f(x)\,dx \\ &= 2\int_{0}^{\infty} f(x)\,dx\end{aligned}$$

(b) Since f is odd, $f(-x) = -f(x)$. Let $u = -x$, $du = -dx$

$$\begin{aligned}\int_{-\infty}^{\infty} f(x)\,dx &= \int_{-\infty}^{0} f(x)\,dx + \int_{\infty}^{0} f(x)\,dx \\ &= \int_{\infty}^{0} f(-u)(-1)\,du + \int_{0}^{\infty} f(x)\,dx \\ &= -\int_{0}^{\infty} f(u)\,du + \int_{0}^{\infty} f(x)\,dx = 0\end{aligned}$$

53. **(a)** $$\int_0^\infty \frac{2x\,dx}{x^2+1} = \lim_{b\to\infty}\int_0^b \frac{2x\,dx}{x^2+1} = \lim_{b\to\infty}\Big[\ln(x^2+1)\Big]_0^b = \lim_{b\to\infty}\ln(b^2+1) = \infty$$

Thus the integral diverges.

(b) Both $\int_0^\infty \frac{2x\,dx}{x^2+1}$ and $\int_{-\infty}^0 \frac{2x\,dx}{x^2+1}$ must converge in order for $\int_{-\infty}^\infty \frac{2x\,dx}{x^2+1}$ to converge.

(c) $$\lim_{b\to\infty}\int_{-b}^b \frac{2x\,dx}{x^2+1} = \lim_{b\to\infty}\Big[\ln(x^2+1)\Big]_{-b}^b = \lim_{b\to\infty}[\ln(b^2+1)-\ln(b^2+1)] = \lim_{b\to\infty}0 = 0.$$

Note that $\frac{2x}{x^2+1}$ is an odd function so $\int_{-b}^b \frac{2x\,dx}{x^2+1} = 0.$

(d) Because the determination of convergence is not made using the method in part (c). In order for the integral to converge, there must be finite areas in both directions (toward ∞ and toward $-\infty$). In this case, there are infinite areas in both directions, but when one computes the integral over an interval $[-b, b]$, there is cancellation which gives 0 as the result.

54. By symmetry, find the perimeter of one side, say for

$0 \le x \le 1, y \ge 0.$

$y^{2/3} = 1 - x^{2/3}$

$y = (1 - x^{2/3})^{3/2}$

$$\frac{dy}{dx} = \frac{3}{2}(1-x^{2/3})^{1/2}\left(-\frac{2}{3}x^{-1/3}\right) = -x^{-1/3}(1-x^{2/3})^{1/2}$$

$$\left(\frac{dy}{dx}\right)^2 = x^{-2/3}(1-x^{2/3}) = (x^{-2/3}-1)$$

$$\sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{x^{-2/3}} = x^{-1/3}$$

$$\int_0^1 x^{-1/3}\,dx = \lim_{b\to0^+}\int_b^1 x^{-1/3}\,dx = \lim_{b\to0^+}\left[\frac{3}{2}x^{2/3}\right]_b^1 = \lim_{b\to0^+}\left[\frac{3}{2}-\frac{3}{2}b^{2/3}\right] = \frac{3}{2}$$

Thus, the perimeter is $4\left(\frac{3}{2}\right) = 6.$

55. Suppose $0 \le f(x) \le g(x)$ for all $x \ge a$.

From the properties of integrals, for any $b > a$, $\int_a^b f(x) \le \int_a^b g(x)\,dx.$

If the infinite integral of g converges, then taking the limit in the above inequality as $b\to\infty$ shows that the infinite integral of f is bounded above by the infinite integral of g. Therefore, the infinite integral of f must be finite and it converges. If the infinite integral of f diverges, it must grow to infinity. So taking the limit in the above inequality as $b\to\infty$ shows that the infinite integral of g must also diverge to infinity.

56. **(a)** For $n = 0$:

$$\int_0^\infty e^{-x}\,dx = \lim_{b\to\infty}\int_0^b e^{-x}\,dx = \lim_{b\to\infty}\Big[-e^{-x}\Big]_0^b = \lim_{b\to\infty}[-e^{-b}+1] = 1$$

For $n = 1$:

$u = x \qquad dv = e^{-x}\,dx$

$du = dx \qquad v = -e^{-x}$

$$\int_0^\infty xe^{-x}\,dx = \lim_{b\to\infty}\int_0^b xe^{-x}\,dx = \lim_{b\to\infty}\left(\Big[-xe^{-x}\Big]_0^b + \int_0^b e^{-x}\,dx\right) = \lim_{b\to\infty}\left(-\frac{b}{e^b}\right) + \lim_{b\to\infty}\int_0^b e^{-x}\,dx = \lim_{b\to\infty}\left(-\frac{1}{e^b}\right) + 1 = 1$$

For $n = 2$:

$u = x^2 \qquad dv = e^{-x}\,dx$

$du = 2x\,dx \qquad v = -e^{-x}$

$$\int_0^\infty x^2e^{-x}\,dx = \lim_{b\to\infty}\int_0^b x^2e^{-x}\,dx = \lim_{b\to\infty}\left(\Big[-x^2e^{-x}\Big]_0^b + \int_0^b 2xe^{-x}\,dx\right) = \lim_{b\to\infty}\left(-\frac{b^2}{e^b}\right) + 2\lim_{b\to\infty}\int_0^b xe^{-x}\,dx = \lim_{b\to\infty}\left(-\frac{2b}{e^b}\right) + 2(1) = \lim_{b\to\infty}\left(-\frac{2}{e^b}\right) + 2 = 2$$

56. continued

(b) Evaluate $\int x^n\, e^{-x}\, dx$ using integration by parts

$u = x^n \qquad dv = e^{-x}\, dx$

$du = nx^{n-1} \qquad v = -e^{-x}$

$$\int x^n\, e^{-x}\, dx = -x^n\, e^{-x} + \int nx^{n-1}\, e^{-x}\, dx$$

$$\begin{aligned} f(n+1) &= \int_0^{\infty} x^n e^{-x}\, dx \\ &= \lim_{b\to\infty} \Big[-x^n e^{-x}\Big]_0^b + \int_0^{\infty} nx^{n-1}e^{-x}\, dx \\ &= \lim_{b\to\infty} \left(-\frac{b^n}{e^b}\right) + n\int_0^{\infty} x^{n-1}e^{-x}\, dx \\ &= nf(n) \end{aligned}$$

$\Big($Note: apply L'Hôpital's Rule n times to show that

$\lim\limits_{b\to\infty} \left(-\dfrac{b^n}{e^b}\right) = 0.\Big)$

(c) Since $f(n+1) = nf(n)$,

$f(n+1) = n(n-1)\cdots f(1) = n!$; thus

$\int_0^{\infty} x^n e^{-x}\, dx$ converges for all integers $n \geq 0$.

57. (a) On a grapher, plot NINT$\left(\dfrac{\sin x}{x}, x, 0, x\right)$ or create a table of values. For large values of x, $f(x)$ appears to approach approximately 1.57.

(b) Yes, the integral appears to converge.

58. (a)
$$\begin{aligned} \int_{-\infty}^{1} \frac{dx}{1+x^2} &= \lim_{b\to-\infty} \int_b^1 \frac{dx}{1+x^2} \\ &= \lim_{b\to-\infty} \Big[\tan^{-1} x\Big]_b^1 \\ &= \lim_{b\to-\infty} (\tan^{-1} 1 - \tan^{-1} b) \\ &= \frac{\pi}{4} + \frac{\pi}{2} = \frac{3\pi}{4} \end{aligned}$$

$$\begin{aligned} \int_1^{\infty} \frac{dx}{1+x^2} &= \lim_{b\to\infty} \int_1^b \frac{dx}{1+x^2} \\ &= \lim_{b\to\infty} \Big[\tan^{-1} x\Big]_1^b \\ &= \lim_{b\to\infty} [\tan^{-1} b - \tan^{-1} 1] \\ &= \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4} \end{aligned}$$

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \frac{3\pi}{4} + \frac{\pi}{4} = \pi$$

(b) $\int_{-\infty}^{c} f(x)\, dx = \int_{-\infty}^{0} f(x)\, dx + \int_0^c f(x)\, dx$

$\int_c^{\infty} f(x)\, dx = \int_c^0 f(x)\, dx + \int_0^{\infty} f(x)\, dx$

Thus,

$$\begin{aligned} &\int_{-\infty}^{c} f(x)\, dx + \int_c^{\infty} f(x)\, dx \\ &= \int_{-\infty}^{0} f(x)\, dx + \int_0^c f(x)\, dx + \int_c^0 f(x)\, dx + \int_0^{\infty} f(x)\, dx \\ &= \int_{-\infty}^{0} f(x)\, dx + \int_0^{\infty} f(x)\, dx, \text{ because} \\ &\int_0^c f(x)\, dx + \int_c^0 f(x)\, dx = \int_0^c f(x)\, dx - \int_0^c f(x)\, dx = 0. \end{aligned}$$

■ Section 8.4 Partial Fractions and Integral Tables (pp. 444–453)

Quick Review 8.4

1. Solving the first equation for B yields $B = -3A - 5$. Substitute into the second equation.

$$\begin{aligned} -2A + 3(-3A - 5) &= 7 \\ -2A - 9A - 15 &= 7 \\ -11A &= 22 \\ A &= -2 \end{aligned}$$

Substituting $A = -2$ into $B = -3A - 5$ gives $B = 1$. The solution is $A = -2$, $B = 1$.

2. Solve by Gaussian elimination. Multiply first equation by -3 and add to second equation. Multiply first equation by -1 and add to third equation.

$$\begin{aligned} A + 2B - C &= 0 \\ -7B + 5C &= 1 \\ -B + 2C &= 4 \end{aligned}$$

Multiply third equation by -7 and add to second equation.

$$\begin{aligned} A + 2B - C &= 0 \\ -9C &= -27 \\ -B + 2C &= 4 \end{aligned}$$

Solve the second equation for C to get $C = 3$. Solve for B by substituting $C = 3$ into the third equation.

$$\begin{aligned} -B + 2(3) &= 4 \\ -B &= -2 \\ B &= 2 \end{aligned}$$

Solve for A by substituting $B = 2$ and $C = 3$ into the first equation.

$$\begin{aligned} A + 2(2) - 3 &= 0 \\ A + 1 &= 0 \\ A &= -1 \end{aligned}$$

The solution is $A = -1$, $B = 2$, $C = 3$.

3.
$$\begin{array}{r} 2x + 1 \\ x^2 - 3x - 4\overline{)2x^3 - 5x^2 - 10x - 7} \\ \underline{2x^3 - 6x^2 \quad - 8x} \\ x^2 \quad - 2x - 7 \\ \underline{x^2 \quad - 3x - 4} \\ x - 3 \end{array}$$

$$2x + 1 + \frac{x-3}{x^2 - 3x - 4}$$

4. $$\begin{array}{r} 2 \\ x^2+4x+5\overline{)\,2x^2+11x+\ \ 6} \\ \underline{2x^2+\ \ 8x+10} \\ 3x-\ \ 4 \end{array}$$

$$2+\frac{3x-4}{x^2+4x+5}$$

5. $$\begin{aligned} x^3-3x^2+x-3 &= x^2(x-3)+(x-3) \\ &= (x-3)(x^2+1) \end{aligned}$$

6. $$\begin{aligned} y^4-5y^2+4 &= (y^2-4)(y^2-1) \\ &= (y-2)(y+2)(y-1)(y+1) \end{aligned}$$

7. $$\begin{aligned} \frac{2}{x+3}-\frac{3}{x-2} &= \frac{2(x-2)}{(x-2)(x+3)}-\frac{3(x+3)}{(x-2)(x+3)} \\ &= \frac{2x-4-3x-9}{(x-2)(x+3)} \\ &= \frac{-x-13}{(x-2)(x+3)} \\ &= -\frac{x+13}{x^2+x-6} \end{aligned}$$

8. $$\begin{aligned} &\frac{x-1}{x^2-4x+5}-\frac{2}{x+5} \\ &= \frac{(x-1)(x+5)}{(x+5)(x^2-4x+5)}-\frac{2(x^2-4x+5)}{(x+5)(x^2-4x+5)} \\ &= \frac{x^2+4x-5-2x^2+8x-10}{(x+5)(x^2-4x+5)} \\ &= \frac{-x^2+12x-15}{(x+5)(x^2-4x+5)} \end{aligned}$$

9. $$\begin{aligned} \frac{t-1}{t^2+2}-\frac{3t+4}{t^2+1} &= \frac{(t-1)(t^2+1)}{(t^2+2)(t^2+1)}-\frac{(3t+4)(t^2+2)}{(t^2+2)(t^2+1)} \\ &= \frac{t^3-t^2+t-1-3t^3-4t^2-6t-8}{(t^2+2)(t^2+1)} \\ &= \frac{-2t^3-5t^2-5t-9}{(t^2+2)(t^2+1)} \\ &= -\frac{2t^3+5t^2+5t+9}{(t^2+2)(t^2+1)} \end{aligned}$$

10. $$\begin{aligned} &\frac{2}{x-1}-\frac{3}{(x-1)^2}+\frac{1}{(x-1)^3} \\ &= \frac{2(x-1)^2}{(x-1)^3}-\frac{3(x-1)}{(x-1)^3}+\frac{1}{(x-1)^3} \\ &= \frac{2x^2-4x+2-3x+3+1}{(x-1)^3} \\ &= \frac{2x^2-7x+6}{(x-1)^3} \end{aligned}$$

Section 8.4 Exercises

1. $$x^2-3x+2=(x-2)(x-1)$$

$$\frac{5x-7}{x^2-3x+2}=\frac{A}{x-2}+\frac{B}{x-1}$$

$$\begin{aligned} 5x-7 &= A(x-1)+B(x-2) \\ &= (A+B)x-(A+2B) \end{aligned}$$

Equating coefficients of like terms gives

$A+B=5$ and $A+2B=7$.

Solving the system simultaneously yields $A=3$, $B=2$.

$$\frac{5x-7}{x^2-3x+2}=\frac{3}{x-2}+\frac{2}{x-1}$$

2. $$x^2-2x+1=(x-1)^2$$

$$\frac{2x+2}{x^2-2x+1}=\frac{A}{x-1}+\frac{B}{(x-1)^2}$$

$$\begin{aligned} 2x+2 &= A(x-1)+B \\ &= Ax+(-A+B) \end{aligned}$$

Equating coefficients of like terms gives

$A=2$ and $-A+B=2$.

Solving the system simultaneously yields $A=2$, $B=4$.

$$\frac{2x+2}{x^2-2x+1}=\frac{2}{x-1}+\frac{4}{(x-1)^2}$$

3. $$\frac{t+1}{t^2(t-1)}=\frac{A}{t-1}+\frac{B}{t}+\frac{C}{t^2}$$

$$\begin{aligned} t+1 &= At^2+Bt(t-1)+C(t-1) \\ &= (A+B)t^2+(-B+C)t-C \end{aligned}$$

Equating coefficients of like terms gives

$A+B=0$, $-B+C=1$, and $-C=1$.

Solving the system simultaneously yields

$A=2$, $B=-2$, $C=-1$.

$$\frac{t+1}{t^2(t-1)}=\frac{2}{t-1}-\frac{2}{t}-\frac{1}{t^2}$$

4. $$s^3-s^2-6s=s(s^2-s-6)=s(s-3)(s+2)$$

$$\frac{4}{s^3-s^2-6s}=\frac{A}{s}+\frac{B}{s-3}+\frac{C}{s+2}$$

$$\begin{aligned} 4 &= A(s-3)(s+2)+B(s)(s+2)+C(s)(s-3) \\ &= A(s^2-s-6)+B(s^2+2s)+C(s^2-3s) \\ &= (A+B+C)s^2+(-A+2B-3C)s-6A \end{aligned}$$

Equating coefficients of like terms gives

$A+B+C=0$, $-A+2B-3C=0$, $-6A=4$

Solving the system simultaneously yields

$A=-\frac{2}{3}$, $B=\frac{4}{15}$, $C=\frac{2}{5}$.

$$\frac{4}{s^3-s^2-6s}=-\frac{2}{3s}+\frac{4}{15(s-3)}+\frac{2}{5(s+2)}$$

5. $$\begin{array}{r} 1 \\ x^2-5x+6\,\overline{)\,x^2 \qquad\quad + 8} \\ \underline{x^2-5x+6} \\ 5x+2 \end{array}$$

$$\frac{x^2+8}{x^2-5x+6} = 1 + \frac{5x+2}{x^2-5x+6}$$

$$x^2-5x+6 = (x-3)(x-2)$$

$$\frac{5x+2}{x^2-5x+6} = \frac{A}{x-3} + \frac{B}{x-2}$$

$$\begin{aligned} 5x+2 &= A(x-2) + B(x-3) \\ &= (A+B)x + (-2A-3B) \end{aligned}$$

Equating coefficients of like terms gives

$A + B = 5$ and $-2A - 3B = 2$

Solving the system simultaneously yields

$A = 17$, $B = -12$.

$$\frac{x^2+8}{x^2-5x+6} = 1 + \frac{17}{x-3} - \frac{12}{x-2}$$

6. $$\begin{array}{r} y \\ y^2+4\,\overline{)\,y^3 \qquad\qquad + 1} \\ \underline{y^3 \qquad + 4y + 0} \\ -4y + 1 \end{array}$$

$$\frac{y^3+1}{y^2+4} = y + \frac{-4y+1}{y^2+4}$$

The rational function cannot be decomposed any further.

7. $1 - x^2 = (1-x)(1+x)$

$$\frac{1}{1-x^2} = \frac{A}{1-x} + \frac{B}{1+x}$$

$$\begin{aligned} 1 &= A(1+x) + B(1-x) \\ &= (A-B)x + (A+B) \end{aligned}$$

Equating coefficients of like terms gives

$A - B = 0$ and $A + B = 1$.

Solving the system simultaneously yields

$A = \frac{1}{2}$, $B = \frac{1}{2}$.

$$\begin{aligned} \int \frac{dx}{1-x^2} &= \int \frac{1/2}{1-x}\,dx + \int \frac{1/2}{1+x}\,dx \\ &= -\frac{1}{2}\ln|1-x| + \frac{1}{2}\ln|1+x| + C \\ &= \frac{1}{2}\ln\left|\frac{1+x}{1-x}\right| + C \end{aligned}$$

8. $x^2 + 2x = x(x+2)$

$$\frac{1}{x^2+2x} = \frac{A}{x} + \frac{B}{x+2}$$

$$\begin{aligned} 1 &= A(x+2) + Bx \\ &= (A+B)x + 2A \end{aligned}$$

Equating coefficients of like terms gives

$A + B = 0$ and $2A = 1$

Solving the system simultaneously yields $A = \frac{1}{2}$, $B = -\frac{1}{2}$.

$$\begin{aligned} \int \frac{dx}{x^2+2x} &= \int \frac{1/2}{x}\,dx + \int \frac{-1/2}{x+2}\,dx \\ &= \frac{1}{2}\ln|x| - \frac{1}{2}\ln|x+2| + C \\ &= \frac{1}{2}\ln\left|\frac{x}{x+2}\right| + C \end{aligned}$$

9. $y^2 - 2y - 3 = (y-3)(y+1)$

$$\frac{y}{y^2-2y-3} = \frac{A}{y-3} + \frac{B}{y+1}$$

$$\begin{aligned} y &= A(y+1) + B(y-3) \\ &= (A+B)y + (A-3B) \end{aligned}$$

Equating coefficients of like terms gives

$A + B = 1$ and $A - 3B = 0$.

Solving the system simultaneously yields $A = \frac{3}{4}$, $B = \frac{1}{4}$.

$$\begin{aligned} \int \frac{y\,dy}{y^2-2y-3} &= \int \frac{3/4}{y-3}\,dy + \int \frac{1/4}{y+1}\,dy \\ &= \frac{3}{4}\ln|y-3| + \frac{1}{4}\ln|y+1| + C \end{aligned}$$

10. $y^2 + y = y(y+1)$

$$\frac{y+4}{y^2+y} = \frac{A}{y} + \frac{B}{y+1}$$

$$\begin{aligned} y+4 &= A(y+1) + By \\ &= (A+B)y + A \end{aligned}$$

Equating coefficients of like terms gives

$A + B = 1$ and $A = 4$.

Solving the system simultaneously yields $A = 4$, $B = -3$.

$$\begin{aligned} \int \frac{y+4}{y^2+y}\,dy &= \int \frac{4}{y}\,dy + \int \frac{-3}{y+1}\,dy \\ &= 4\ln|y| - 3\ln|y+1| + C \end{aligned}$$

11. $t^3 + t^2 - 2t = t(t^2 + t - 2) = t(t + 2)(t - 1)$

$$\frac{1}{t^3 + t^2 - 2t} = \frac{A}{t} + \frac{B}{t+2} + \frac{C}{t-1}$$

$$\begin{aligned} 1 &= A(t + 2)(t - 1) + B(t)(t - 1) + C(t)(t + 2) \\ &= A(t^2 + t - 2) + B(t^2 - t) + C(t^2 + 2t) \\ &= (A + B + C)t^2 + (A - B + 2C)t - 2A \end{aligned}$$

Equating coefficients of like terms gives

$A + B + C = 0$, $A - B + 2C = 0$, and $-2A = 1$.

Solving the system simultaneously yields

$A = -\frac{1}{2}, B = \frac{1}{6}, C = \frac{1}{3}.$

$$\begin{aligned} \int \frac{dt}{t^3 + t^2 - 2t} &= \int \frac{-1/2}{t}\,dt + \int \frac{1/6}{t+2}\,dt + \int \frac{1/3}{t-1}\,dt \\ &= -\frac{1}{2}\ln|t| + \frac{1}{6}\ln|t + 2| + \frac{1}{3}\ln|t - 1| + C \end{aligned}$$

12. $2t^3 - 8t = 2t(t^2 - 4) = 2t(t - 2)(t + 2)$

$$\frac{t+3}{2t^3 - 8t} = \frac{A}{t} + \frac{B}{t-2} + \frac{C}{t+2}$$

$$\begin{aligned} t + 3 &= 2A(t - 2)(t + 2) + 2B(t)(t + 2) + 2C(t)(t - 2) \\ &= 2A(t^2 - 4) + 2B(t^2 + 2t) + 2C(t^2 - 2t) \\ &= (2A + 2B + 2C)t^2 + (4B - 4C)t - 8A \end{aligned}$$

Equating coefficients of like terms gives

$2A + 2B + 2C = 0$, $4B - 4C = 1$, $-8A = 3$

Solving the system simultaneously yields

$A = -\frac{3}{8}, B = \frac{5}{16}, C = \frac{1}{16}.$

$$\begin{aligned} \int \frac{t+3}{2t^3 - 8t}\,dt &= \int \frac{-3/8}{t}\,dt + \int \frac{5/16}{t-2}\,dt + \int \frac{1/16}{t+2}\,dt \\ &= -\frac{3}{8}\ln|t| + \frac{5}{16}\ln|t - 2| + \frac{1}{16}\ln|t + 2| + C \end{aligned}$$

13.

$$\begin{array}{r} s \\ s^2 + 4\overline{)s^3} \\ \underline{s^3 + 4s} \\ -4s \end{array}$$

$$\frac{s^3}{s^2 + 4} = s + \frac{-4s}{s^2 + 4}$$

$$\begin{aligned} \int \frac{s^3}{s^2 + 4}\,ds &= \int s\,ds - \int \frac{4s}{s^2 + 4}\,ds \\ &= \frac{1}{2}s^2 - 2\ln(s^2 + 4) + C \end{aligned}$$

14.

$$\begin{array}{r} s^2 - 1 \\ s^2 + 1\overline{)s^4 + 2s} \\ \underline{s^4 + s^2} \\ -s^2 + 2s \\ \underline{-s^2 - 1} \\ 2s + 1 \end{array}$$

$$\frac{s^4 + 2s}{s^2 + 1} = s^2 - 1 + \frac{2s + 1}{s^2 + 1} = s^2 - 1 + \frac{2s}{s^2 + 1} + \frac{1}{s^2 + 1}$$

$$\begin{aligned} \int \frac{s^4 + 2s}{s^2 + 1}\,ds &= \int (s^2 - 1)\,ds + \int \frac{2s\,ds}{s^2 + 1} + \int \frac{ds}{s^2 + 1} \\ &= \frac{1}{3}s^3 - s + \ln(s^2 + 1) + \tan^{-1} s + C \end{aligned}$$

15.

$$\begin{array}{r} 5 \\ x^2 + x + 1\overline{)5x^2} \\ \underline{5x^2 + 5x + 5} \\ -5x - 5 \end{array}$$

$$\frac{5x^2}{x^2 + x + 1} = 5 - \frac{5x + 5}{x^2 + x + 1}$$

$$\begin{aligned} \int \frac{5x^2\,dx}{x^2 + x + 1} &= \int 5\,dx - 5\int \frac{x + 1}{x^2 + x + 1}\,dx \\ &= 5x - 5\int \frac{x + 1}{x^2 + x + 1}\,dx \end{aligned}$$

To evaluate the second integral, complete the square in the denominator.

$$x^2 + x + 1 = x^2 + x + \frac{1}{4} + \frac{3}{4} = \left(x + \frac{1}{2}\right)^2 + \frac{3}{4}$$

$$\begin{aligned} &\int \frac{x + 1}{x^2 + x + 1}\,dx \\ &= \int \frac{x + 1}{(x + 1/2)^2 + 3/4}\,dx \\ &= \int \frac{x + 1/2}{(x + 1/2)^2 + 3/4}\,dx + \int \frac{1/2}{(x + 1/2)^2 + 3/4}\,dx \\ &= \frac{1}{2}\ln\left[\left(x + \frac{1}{2}\right)^2 + \frac{3}{4}\right] + \frac{1}{2}\int \frac{dx}{(x + 1/2)^2 + (\sqrt{3}/2)^2} \\ &= \frac{1}{2}\ln(x^2 + x + 1) + \frac{1}{2}\left(\frac{2}{\sqrt{3}}\right)\tan^{-1}\left(\frac{x + 1/2}{\sqrt{3}/2}\right) \\ &= \frac{1}{2}\ln(x^2 + x + 1) + \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x + 1}{\sqrt{3}}\right) \end{aligned}$$

The second integral was evaluated by using Formula 16 from the Brief Table of Integrals.

$$\begin{aligned} &\int \frac{5x^2\,dx}{x^2 + x + 1} \\ &= 5x - \frac{5}{2}\ln(x^2 + x + 1) - \frac{5}{\sqrt{3}}\tan^{-1}\left(\frac{2x + 1}{\sqrt{3}}\right) + C \end{aligned}$$

16. $(x - 1)(x^2 + 2x + 1) = (x - 1)(x + 1)^2$

$$\frac{x^2}{(x - 1)(x^2 + 2x + 1)} = \frac{A}{x - 1} + \frac{B}{x + 1} + \frac{C}{(x + 1)^2}$$

$$\begin{aligned} x^2 &= A(x + 1)^2 + B(x - 1)(x + 1) + C(x - 1) \\ &= A(x^2 + 2x + 1) + B(x^2 - 1) + C(x - 1) \\ &= (A + B)x^2 + (2A + C)x + A - B - C \end{aligned}$$

Equating coefficients of like terms gives

$A + B = 1$, $2A + C = 0$, and $A - B - C = 0$.

Solving the system simultaneously yields

$A = \frac{1}{4}, B = \frac{3}{4}, C = -\frac{1}{2}.$

$$\begin{aligned} &\int \frac{x^2\,dx}{(x - 1)(x^2 + 2x + 1)} \\ &= \int \frac{1/4}{x - 1}\,dx + \int \frac{3/4}{x + 1}\,dx + \int \frac{-1/2}{(x + 1)^2}\,dx \\ &= \frac{1}{4}\ln|x - 1| + \frac{3}{4}\ln|x + 1| + \frac{1}{2(x + 1)} + C \end{aligned}$$

17. $(x^2-1)^2 = (x+1)^2(x-1)^2$

$$\frac{1}{(x^2-1)^2} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x-1} + \frac{D}{(x-1)^2}$$

$$\begin{aligned} 1 &= A(x+1)(x-1)^2 + B(x-1)^2 + C(x+1)^2(x-1) \\ &\quad + D(x+1)^2 \\ &= A(x^3-x^2-x+1) + B(x^2-2x+1) \\ &\quad + C(x^3+x^2-x-1) + D(x^2+2x+1) \\ &= (A+C)x^3 + (-A+B+C+D)x^2 \\ &\quad + (-A-2B-C+2D)x + (A+B-C+D) \end{aligned}$$

Equating coefficients of like terms gives

$A+C=0$, $-A+B+C+D=0$,

$-A-2B-C+2D=0$, and $A+B-C+D=1$

Solving the system simultaneously yields

$A = \frac{1}{4}$, $B = \frac{1}{4}$, $C = -\frac{1}{4}$, $D = \frac{1}{4}$.

$$\begin{aligned} &\int \frac{dx}{(x^2-1)^2} \\ &= \int \frac{1/4}{x+1}\,dx + \int \frac{1/4}{(x+1)^2}\,dx + \int \frac{-1/4}{x-1}\,dx + \int \frac{1/4}{(x-1)^2}\,dx \\ &= \frac{1}{4}\ln|x+1| - \frac{1}{4(x+1)} - \frac{1}{4}\ln|x-1| - \frac{1}{4(x-1)} + C \end{aligned}$$

18. $x^2+5x-6 = (x+6)(x-1)$

$$\frac{x+4}{x^2+5x-6} = \frac{A}{x+6} + \frac{B}{x-1}$$

$$\begin{aligned} x+4 &= A(x-1) + B(x+6) \\ &= (A+B)x + (-A+6B) \end{aligned}$$

Equating coefficients of like terms gives

$A+B=1$ and $-A+6B=4$.

Solving the system simultaneously yields

$A = \frac{2}{7}$, $B = \frac{5}{7}$.

$$\begin{aligned} \int \frac{x+4}{x^2+5x-6}\,dx &= \int \frac{2/7}{x+6}\,dx + \int \frac{5/7}{x-1}\,dx \\ &= \frac{2}{7}\ln|x+6| + \frac{5}{7}\ln|x-1| + C \end{aligned}$$

19. Complete the square in the denominator.

$r^2-2r+2 = r^2-2r+1+1 = (r-1)^2+1$

$$\int \frac{2\,dr}{r^2-2r+2} = \int \frac{2\,dr}{(r-1)^2+1} = 2\tan^{-1}(r-1) + C$$

20. Complete the square in the denominator.

$r^2-4r+5 = r^2-4r+4+1 = (r-2)^2+1$

$$\int \frac{3\,dr}{r^2-4r+5} = \int \frac{3\,dr}{(r-2)^2+1} = 3\tan^{-1}(r-2) + C$$

21. $x^3-1 = (x-1)(x^2+x+1)$

$$\frac{x^2-2x-2}{x^3-1} = \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1}$$

$$\begin{aligned} x^2-2x-2 &= A(x^2+x+1) + (Bx+C)(x-1) \\ &= (A+B)x^2 + (A-B+C)x + (A-C) \end{aligned}$$

Equating coefficients of like terms gives

$A+B=1$, $A-B+C=-2$, and $A-C=-2$.

Solving the system simultaneously yields

$A=-1$, $B=2$, $C=1$.

$$\begin{aligned} \int \frac{x^2-2x-2}{x^3-1}\,dx &= \int \frac{-1}{x-1}\,dx + \int \frac{2x+1}{x^2+x+1}\,dx \\ &= -\ln|x-1| + \ln(x^2+x+1) + C \end{aligned}$$

22. $x^3+1 = (x+1)(x^2-x+1)$

$$\frac{x^2-4x+4}{x^3+1} = \frac{A}{x+1} + \frac{Bx+C}{x^2-x+1}$$

$$\begin{aligned} x^2-4x+4 &= A(x^2-x+1) + (Bx+C)(x+1) \\ &= (A+B)x^2 + (-A+B+C)x + (A+C) \end{aligned}$$

Equating coefficients of like terms gives

$A+B=1$, $-A+B+C=-4$, and $A+C=4$.

Solving the system simultaneously yields

$A=3$, $B=-2$, $C=1$.

$$\begin{aligned} \int \frac{x^2-4x+4}{x^3+1}\,dx &= \int \frac{3}{x+1}\,dx + \int \frac{-2x+1}{x^2-x+1}\,dx \\ &= 3\ln|x+1| - \ln(x^2-x+1) + C \end{aligned}$$

23. $\dfrac{3x^2-2x+12}{(x^2+4)^2} = \dfrac{Ax+B}{x^2+4} + \dfrac{Cx+D}{(x^2+4)^2}$

$$\begin{aligned} 3x^2-2x+12 &= (Ax+B)(x^2+4) + (Cx+D) \\ &= Ax^3 + Bx^2 + (4A+C)x + 4B + D \end{aligned}$$

Equating coefficients of like terms gives

$A=0$, $B=3$, $4A+C=-2$, and $4B+D=12$

Solving the system simultaneously yields

$A=0$, $B=3$, $C=-2$, $D=0$.

$$\begin{aligned} \int \frac{3x^2-2x+12}{(x^2+4)^2}\,dx &= \int \frac{3}{x^2+4}\,dx + \int \frac{-2x}{(x^2+4)^2}\,dx \\ &= \frac{3}{2}\tan^{-1}\frac{x}{2} + \frac{1}{x^2+4} + C \end{aligned}$$

The first integral was evaluated by using Formula 16 from the Brief Table of Integrals.

24. $\dfrac{x^3+2x^2+2}{(x^2+1)^2}=\dfrac{Ax+B}{(x^2+1)}+\dfrac{Cx+D}{(x^2+1)^2}$

$$x^3+2x^2+2=(Ax+B)(x^2+1)+(Cx+D)$$

$$=Ax^3+Bx^2+(A+C)x+(B+D)$$

Equating coefficients of like terms gives

$A=1, B=2, A+C=0, B+D=2$

Solving the system simultaneously yields

$A=1, B=2, C=-1, D=0.$

$$\int\frac{x^3+2x^2+2}{(x^2+1)^2}\,dx$$

$$=\int\frac{x+2}{x^2+1}\,dx+\int\frac{-x}{(x^2+1)^2}\,dx$$

$$=\int\frac{x}{x^2+1}\,dx+\int\frac{2}{x^2+1}\,dx-\int\frac{x}{(x^2+1)^2}\,dx$$

$$=\frac{1}{2}\ln(x^2+1)+2\tan^{-1}x+\frac{1}{2(x^2+1)}+C$$

25. $$\begin{array}{r} 1 \\ \theta+1\overline{)\theta} \\ \underline{\theta+1} \\ -1 \end{array}$$

$$\frac{\theta}{\theta+1}=1-\frac{1}{\theta+1}$$

$$\int_0^1\frac{\theta}{\theta+1}\,d\theta=\int_0^1 d\theta-\int_0^1\frac{1}{\theta+1}\,d\theta$$

$$=\Big[\theta\Big]_0^1-\Big[\ln|\theta+1|\Big]_0^1$$

$$=1-\ln 2$$

26. $$\begin{array}{r} 1 \\ \theta^2+1\overline{)\theta^2} \\ \underline{\theta^2+1} \\ -1 \end{array}$$

$$\frac{\theta^2}{\theta^2+1}=1-\frac{1}{\theta^2+1}$$

$$\int_0^2\frac{\theta^2}{\theta^2+1}\,d\theta=\int_0^2 d\theta-\int_0^2\frac{1}{\theta^2+1}\,d\theta$$

$$=\Big[\theta\Big]_0^2-\Big[\tan^{-1}\theta\Big]_0^2$$

$$=2-\tan^{-1}2$$

27. $\dfrac{1}{y^2-y}\,dy=e^x\,dx$

$$\int\frac{1}{y(y-1)}\,dy=\int e^x\,dx=e^x+C$$

$$\frac{1}{y(y-1)}=\frac{A}{y}+\frac{B}{y-1}$$

$$1=A(y-1)+B(y)$$

$$=(A+B)y-A$$

Equating coefficients of like terms gives

$A+B=0$ and $-A=1$

Solving the system simultaneously yields $A=-1, B=1.$

$$\int\frac{1}{y(y-1)}\,dy=\int-\frac{1}{y}\,dy+\int\frac{1}{y-1}\,dy$$

$$=-\ln|y|+\ln|y-1|+C_2$$

$-\ln|y|+\ln|y-1|=e^x+C$

Substitute $x=0, y=2.$

$-\ln 2+0=1+C$ or $C=-1-\ln 2$

The solution to the initial value problem is

$-\ln|y|+\ln|y-1|=e^x-1-\ln 2.$

28. $\dfrac{1}{(y+1)^2}\,dy=\sin\theta\,d\theta$

$$\int\frac{1}{(y+1)^2}\,dy=\int\sin\theta\,d\theta$$

$$-\frac{1}{y+1}=-\cos\theta+C$$

Substitute $x=\dfrac{\pi}{2}, y=0.$

$-1=0+C$ or $C=-1$

The solution to the initial value problem is

$$-\frac{1}{y+1}=-\cos\theta-1$$

$$y+1=\frac{1}{\cos\theta+1}$$

$$y=\frac{1}{\cos\theta+1}-1$$

29. $dy=\dfrac{dx}{x^2-3x+2}$

$x^2-3x+2=(x-2)(x-1)$

$$\frac{1}{x^2-3x+2}=\frac{A}{x-2}+\frac{B}{x-1}$$

$1=A(x-1)+B(x-2)$

$1=(A+B)x-A-2B$

Equating coefficients of like terms gives

$A+B=0, -A-2B=1$

Solving the system simultaneously yields $A=1, B=-1.$

$$\int dy=\int\frac{dx}{x^2-3x+2}=\int\frac{dx}{x-2}-\int\frac{dx}{x-1}$$

$y=\ln|x-2|-\ln|x-1|+C$

Substitute $x=3, y=0.$

$0=0-\ln 2+C$ or $C=\ln 2$

The solution to the initial value problem is

$y=\ln|x-2|-\ln|x-1|+\ln 2$

30. $$\frac{ds}{2s+2}=\frac{dt}{t^2+2t}$$

$$\int\frac{ds}{2s+2}=\frac{1}{2}\int\frac{ds}{s+1}=\frac{1}{2}\ln|s+1|+C_1$$

$$t^2+2t=t(t+2)$$

$$\begin{aligned}\frac{1}{t^2+2t}&=\frac{A}{t}+\frac{B}{t+2}\\ 1&=A(t+2)+Bt\\ 1&=(A+B)t+2A\end{aligned}$$

Equating coefficients of like terms gives

$A+B=0$ and $2A=1$

Solving the system simultaneously yields $A=\frac{1}{2}$, $B=-\frac{1}{2}$.

$$\begin{aligned}\int\frac{dt}{t^2+2t}&=\int\frac{1/2}{t}\,dt-\int\frac{1/2}{t+2}\,dt\\ &=\frac{1}{2}\ln|t|-\frac{1}{2}\ln|t+2|+C_2\end{aligned}$$

$$\frac{1}{2}\ln|s+1|=\frac{1}{2}\ln|t|-\frac{1}{2}\ln|t+2|+C_3$$

$$\ln|s+1|=\ln|t|-\ln|t+2|+C$$

Substitute $t=1$, $s=1$.

$\ln 2=0-\ln 3+C$ or $C=\ln 2+\ln 3=\ln 6$

The solution to the initial value problem is

$$\ln|s+1|=\ln|t|-\ln|t+2|+\ln 6$$

$$\ln|s+1|=\ln\left|\frac{6t}{t+2}\right|$$

$$|s+1|=\left|\frac{6t}{t+2}\right|.$$

31. (a) Complete the square in the denominator.

$$\begin{aligned}5+4x-x^2&=5-(x^2-4x)\\ &=9-(x^2-4x+4)\\ &=9-(x-2)^2\end{aligned}$$

Let $u=x-2$ so $du=dx$, and then use Formula 18 with $x=u$ and $a=3$.

$$\begin{aligned}\int\frac{dx}{5+4x-x^2}&=\int\frac{du}{9-u^2}\\ &=\frac{1}{6}\ln\left|\frac{u+3}{u-3}\right|+C\\ &=\frac{1}{6}\ln\left|\frac{x+1}{x-5}\right|+C\end{aligned}$$

(b) $$\begin{aligned}&\frac{d}{dx}\left(\frac{1}{2a}\ln\left|\frac{x+a}{x-a}\right|+C\right)\\ &=\frac{1}{2a}\frac{d}{dx}\left(\ln|x+a|-\ln|x-a|\right)\\ &=\frac{1}{2a}\left(\frac{1}{x+a}-\frac{1}{x-a}\right)\\ &=\frac{1}{2a}\left[\frac{x-a}{(x+a)(x-a)}-\frac{x+a}{(x+a)(x-a)}\right]\\ &=-\frac{1}{x^2-a^2}=\frac{1}{a^2-x^2}\end{aligned}$$

32. (a) Complete the square in the denominator.

$$\begin{aligned}x^2-2x+2&=(x^2-2x+1)+1\\ &=(x-1)^2+1\\ &=1+(x-1)^2\end{aligned}$$

Let $u=x-1$ so $du=dx$ and then use Formula 17 with $x=u$ and $a=1$.

$$\begin{aligned}\int\frac{dx}{(x^2-2x+2)^2}&=\int\frac{du}{(1+u^2)^2}\\ &=\frac{u}{2(1+u^2)}+\frac{1}{2}\tan^{-1}u+C\\ &=\frac{x-1}{2(x^2-2x+2)}+\frac{1}{2}\tan^{-1}(x-1)+C\end{aligned}$$

(b) $$\begin{aligned}&\frac{d}{dx}\left[\frac{x}{2a^2(a^2+x^2)}+\frac{1}{2a^3}\tan^{-1}\frac{x}{a}+C\right]\\ &=\frac{1}{2a^2}\left[\frac{(a^2+x^2)-x(2x)]}{(a^2+x^2)^2}\right]+\frac{1}{2a^3}\left[\frac{1/a}{1+(x/a)^2}\right]\\ &=\frac{a^2-x^2}{2a^2(a^2+x^2)^2}+\frac{1}{2a^4}\left(\frac{a^2}{a^2+x^2}\right)\\ &=\frac{1}{2a^2}\left[\frac{a^2-x^2}{(a^2+x^2)^2}+\frac{1}{a^2+x^2}\right]\\ &=\frac{1}{2a^2}\left[\frac{2a^2}{(a^2+x^2)^2}\right]=\frac{1}{(a^2+x^2)^2}\end{aligned}$$

33. Volume $=\displaystyle\int_{0.5}^{2.5}\pi\left(\frac{9}{3x-x^2}\right)dx=9\pi\int_{0.5}^{2.5}\frac{dx}{3x-x^2}$

$$3x-x^2=x(3-x)$$

$$\begin{aligned}\frac{1}{3x-x^2}&=\frac{A}{x}+\frac{B}{3-x}\\ 1&=A(3-x)+Bx\\ &=(-A+B)x+3A\end{aligned}$$

Equating coefficients of like terms gives

$-A+B=0$ and $3A=1$

Solving the system simultaneously yields $A=\frac{1}{3}$, $B=\frac{1}{3}$.

$$\begin{aligned}9\pi\int_{0.5}^{2.5}\frac{dx}{3x-x^2}&=3\pi\left(\int_{0.5}^{2.5}\frac{dx}{x}+\int_{0.5}^{2.5}\frac{dx}{3-x}\right)\\ &=3\pi\left(\Big[\ln|x|\Big]_{0.5}^{2.5}+\Big[-\ln|3-x|\Big]_{0.5}^{2.5}\right)\\ &=3\pi(\ln 2.5-\ln 0.5-\ln 0.5+\ln 2.5)\\ &=3\pi\ln 25=6\pi\ln 5\end{aligned}$$

34. Volume $= \int_0^1 2\pi x\left[\frac{2}{(x+1)(2-x)}\right] dx$

$$= 4\pi\int_0^1 \frac{x\,dx}{(x+1)(2-x)}$$

$$\frac{x}{(x+1)(2-x)} = \frac{A}{x+1} + \frac{B}{2-x}$$

$$x = A(2-x) + B(x+1)$$

$$= (-A+B)x + (2A+B)$$

Equating coefficients of like terms gives

$-A + B = 1$ and $2A + B = 0$

Solving the system simultaneously yields $A = -\frac{1}{3}$, $B = \frac{2}{3}$.

$$4\pi\int_0^1 \frac{x\,dx}{(x+1)(2-x)}$$

$$= \frac{4\pi}{3}\left(\int_0^1 \frac{-dx}{x+1} + \int_0^1 \frac{2\,dx}{2-x}\right)$$

$$= \frac{4\pi}{3}\left(\Big[-\ln|x+1|\Big]_0^1 + \Big[-2\ln|2-x|\Big]_0^1\right)$$

$$= \frac{4\pi}{3}(-\ln 2 + 2\ln 2) = \frac{4\pi\ln 2}{3}$$

35. $y = 3\tan\theta,\ dy = 3\sec^2\theta\,d\theta,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$9 + y^2 = 9 + 9\tan^2\theta = 9\sec^2\theta$

$$\int\frac{dy}{\sqrt{9+y^2}} = \int\frac{3\sec^2\theta}{|3\sec\theta|}\,d\theta$$

$$= \int\sec\theta\,d\theta$$

$$= \ln|\sec\theta + \tan\theta| + C_1$$

$$= \ln\left|\frac{\sqrt{9+y^2}}{3} + \frac{y}{3}\right| + C_1$$

$$= \ln\left|\sqrt{9+y^2} + y\right| - \ln 3 + C_1$$

$$= \ln\left|\sqrt{9+y^2} + y\right| + C$$

Use Formula 88 for $\int\sec\theta\,d\theta$ with $x = \theta$ and $a = 1$. Use Figure 8.18(a) from the text with $a = 3$ to get

$\sqrt{9+y^2} = 3|\sec\theta|$.

36. $t = 5\sin\theta,\ dt = 5\cos\theta\,d\theta,\ -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

$25 - t^2 = 25 - 25\sin^2\theta = 25\cos^2\theta$

$$\int\sqrt{25-t^2}\,dt = \int 5|\cos\theta|\,(5\cos\theta)\,d\theta$$

$$= 25\int\cos^2\theta\,d\theta$$

$$= 25\int\frac{1+\cos 2\theta}{2}\,d\theta$$

$$= \frac{25\theta}{2} + \frac{25\sin 2\theta}{4} + C$$

$$= \frac{25}{2}\theta + \frac{25}{2}\sin\theta\cos\theta + C$$

$$= \frac{25}{2}\sin^{-1}\frac{t}{5} + \frac{25}{2}\left(\frac{t}{5}\right)\frac{\sqrt{25-t^2}}{5} + C$$

$$= \frac{25}{2}\sin^{-1}\frac{t}{5} + \frac{t\sqrt{25-t^2}}{2} + C$$

Use Figure 8.18(b) from the text with $a = 5$ and $x = t$ to get $\sqrt{25-t^2} = 5|\cos\theta|$.

37. $x = \frac{7}{2}\sec\theta,\ dx = \frac{7}{2}\sec\theta\tan\theta\,d\theta,\ 0 \le \theta < \frac{\pi}{2}$

$4x^2 - 49 = 49\sec^2\theta - 49 = 49\tan^2\theta$

$$\int\frac{dx}{\sqrt{4x^2-49}} = \int\frac{7/2\sec\theta\tan\theta\,d\theta}{|7\tan\theta|}$$

$$= \int\frac{1}{2}\sec\theta\,d\theta$$

$$= \frac{1}{2}\ln|\sec\theta + \tan\theta| + C_1$$

$$= \frac{1}{2}\ln\left|\frac{2x}{7} + \frac{\sqrt{4x^2-49}}{7}\right| + C_1$$

$$= \frac{1}{2}\ln\left|2x + \sqrt{4x^2-49}\right| - \frac{1}{2}\ln 7 + C_1$$

$$= \frac{1}{2}\ln\left|2x + \sqrt{4x^2-49}\right| + C$$

Use Figure 8.18(c) from the text with $a = \frac{7}{2}$ to get

$\sqrt{x^2 - \frac{49}{4}} = \frac{7}{2}\tan\theta$.

38. $x = \tan\theta$, $dx = \sec^2\theta\, d\theta$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$x^2 + 1 = \tan^2\theta + 1 = \sec^2\theta$

$$\begin{aligned}
\int \frac{dx}{x^2\sqrt{x^2+1}} &= \int \frac{\sec^2\theta\, d\theta}{\tan^2\theta\, |\sec\theta|} \\
&= \int \frac{\sec\theta\, d\theta}{\tan^2\theta} \\
&= \int \frac{\cos\theta}{\sin^2\theta}\, d\theta \\
&= \int (\sin\theta)^{-2}\cos\theta\, d\theta \\
&= -(\sin\theta)^{-1} + C \\
&= -\csc\theta + C \\
&= -\frac{\sqrt{1+x^2}}{x} + C
\end{aligned}$$

Use Figure 8.18(a) from the text with $a = 1$ to get $\csc\theta = \frac{\sqrt{1+x^2}}{x}$.

39. $x = \sin\theta$, $dx = \cos\theta\, d\theta$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$1 - x^2 = 1 - \sin^2\theta = \cos^2\theta$

$$\begin{aligned}
\int \frac{x^3\, dx}{\sqrt{1-x^2}} &= \int \frac{\sin^3\theta\cos\theta\, d\theta}{|\cos\theta|} \\
&= \int \sin^3\theta\, d\theta \\
&= \int (1 - \cos^2\theta)\sin\theta\, d\theta \\
&= -\cos\theta + \frac{1}{3}\cos^3\theta + C \\
&= \cos\theta\left(-1 + \frac{1}{3}\cos^2\theta\right) + C \\
&= \sqrt{1-x^2}\left[-1 + \frac{1}{3}(1-x^2)\right] + C \\
&= -\frac{x^2\sqrt{1-x^2}}{3} - \frac{2\sqrt{1-x^2}}{3} + C
\end{aligned}$$

Use Figure 18.8(b) from the text with $a = 1$

to get $\sqrt{1-x^2} = \cos\theta$.

40. $x = \sec\theta$, $dx = \sec\theta\tan\theta\, d\theta$, $0 \le \theta < \frac{\pi}{2}$

$x^2 - 1 = \sec^2\theta - 1 = \tan^2\theta$

$$\begin{aligned}
\int \frac{2\, dx}{x^3\sqrt{x^2-1}} &= \int \frac{2\sec\theta\tan\theta\, d\theta}{\sec^3\theta\, |\tan\theta|} \\
&= \int \frac{2\, d\theta}{\sec^2\theta} \\
&= \int 2\cos^2\theta\, d\theta \\
&= \int \left(1 + \cos 2\theta\right) d\theta \\
&= \theta + \frac{1}{2}\sin 2\theta + C \\
&= \theta + \sin\theta\cos\theta + C \\
&= \sec^{-1} x + \frac{\sqrt{x^2-1}}{x}\left(\frac{1}{x}\right) + C \\
&= \sec^{-1} x + \frac{\sqrt{x^2-1}}{x^2} + C
\end{aligned}$$

41. $z = 4\sin\theta$, $dz = 4\cos\theta\, d\theta$, $0 < \theta < \frac{\pi}{2}$

$16 - z^2 = 16 - 16\sin^2\theta = 16\cos^2\theta$

$$\begin{aligned}
&\int \frac{\sqrt{16-z^2}}{z}\, dz \\
&= \int \frac{|4\cos\theta|\,(4\cos\theta)\, d\theta}{4\sin\theta} \\
&= \int \frac{4\cos^2\theta}{\sin\theta}\, d\theta \\
&= \int \frac{4 - 4\sin^2\theta}{\sin\theta}\, d\theta \\
&= \int (4\csc\theta - 4\sin\theta)\, d\theta \\
&= -4\ln|\csc\theta + \cot\theta| + 4\cos\theta + C \\
&= -4\ln\left|\frac{4}{z} + \frac{\sqrt{16-z^2}}{z}\right| + 4\left(\frac{\sqrt{16-z^2}}{4}\right) + C \\
&= -4\ln\left|\frac{4 + \sqrt{16-z^2}}{z}\right| + \sqrt{16-z^2} + C
\end{aligned}$$

Use Formula 89 with $a = 1$ and $x = \theta$. Use Figure 8.18(b)

from the text with $a = 4$ to get

$\csc\theta = \frac{4}{|z|}$, $\cot\theta = \frac{\sqrt{16-z^2}}{|z|}$ and $\cos\theta = \frac{\sqrt{16-z^2}}{4}$.

42. $w = 2\sin\theta$, $dw = 2\cos\theta\, d\theta$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$4 - w^2 = 4 - 4\sin^2\theta = 4\cos^2\theta$

$$\begin{aligned}
\int \frac{8\, dw}{w^2\sqrt{4-w^2}} &= \int \frac{8(2\cos\theta)\, d\theta}{4\sin^2\theta\, |2\cos\theta|} \\
&= \int 2\csc^2\theta\, d\theta \\
&= -2\cot\theta + C \\
&= \frac{-2\sqrt{4-w^2}}{w} + C
\end{aligned}$$

Use Figure 8.18(b) from the text with $a = 2$ and $x = w$ to get $\cot\theta = \frac{\sqrt{4-w^2}}{w}$.

43. $dy = \dfrac{dx}{\sqrt{x^2 - 9}}$

$x = 3\sec\theta,\ dx = 3\sec\theta\tan\theta\,d\theta,\ 0 < \theta < \dfrac{\pi}{2}$

$x^2 - 9 = 9\sec^2\theta - 9 = 9\tan^2\theta$

$$\begin{aligned} y &= \int \frac{dx}{\sqrt{x^2-9}} \\ &= \int \frac{3\sec\theta\tan\theta\,d\theta}{|3\tan\theta|} \\ &= \int \sec\theta\,d\theta \\ &= \ln|\sec\theta + \tan\theta| + C \\ &= \ln\left|\frac{x}{3} + \frac{\sqrt{x^2-9}}{3}\right| + C \end{aligned}$$

Substitute $x = 5$, $y = \ln 3$.

$\ln 3 = \ln\left(\dfrac{5}{3} + \dfrac{4}{3}\right) + C$ or $C = 0$

The solution to the initial value problem is

$$y = \ln\left|\frac{x}{3} + \frac{\sqrt{x^2-9}}{3}\right|.$$

44. $(x^2 + 1)^2\dfrac{dy}{dx} = \sqrt{x^2 + 1}$

$dy = \dfrac{dx}{(x^2+1)^{3/2}}$

$x = \tan\theta,\ dx = \sec^2\theta\,d\theta,\ -\dfrac{\pi}{2} < \theta < \dfrac{\pi}{2}$

$x^2 + 1 = \tan^2\theta + 1 = \sec^2\theta$

$$\begin{aligned} y &= \int \frac{dx}{(x^2+1)^{3/2}} \\ &= \int \frac{\sec^2\theta\,d\theta}{|\sec^3\theta|} \\ &= \int \cos\theta\,d\theta \\ &= \sin\theta + C \\ &= \frac{x}{\sqrt{x^2+1}} + C \end{aligned}$$

Substitute $x = 0$, $y = 1$.

$1 = C$

The solution to the initial value problem is

$$y = \frac{x}{\sqrt{x^2+1}} + 1.$$

45. For $x \ge 0$, $y \ge 0$ on $[0, 3]$

$\text{Area} = \displaystyle\int_0^3 \frac{\sqrt{9-x^2}}{3}\,dx$

$x = 3\sin\theta,\ dx = 3\cos\theta\,d\theta,\ 0 \le \theta \le \dfrac{\pi}{2}$

$9 - x^2 = 9 - 9\sin^2\theta = 9\cos^2\theta$

When $x = 0$, $\theta = 0$ and when $x = 3$, $\theta = \dfrac{\pi}{2}$.

$$\begin{aligned} \int_0^3 \frac{\sqrt{9-x^2}}{3}\,dx &= \int_0^{\pi/2} \frac{3|\cos\theta|}{3}\,3\cos\theta\,d\theta \\ &= \int_0^{\pi/2} 3\cos^2\theta\,d\theta \\ &= 3\int_0^{\pi/2} \frac{1+\cos 2\theta}{2}\,d\theta \\ &= 3\left[\frac{\theta}{2} + \frac{\sin 2\theta}{4}\right]_0^{\pi/2} \\ &= \frac{3\pi}{4} \approx 2.356 \end{aligned}$$

46. $\text{Volume} = \displaystyle\int_0^1 \pi\left(\frac{2}{1+x^2}\right)^2 dx = 4\pi\int_0^1 \frac{dx}{(1+x^2)^2}$

$x = \tan\theta,\ dx = \sec^2\theta\,d\theta,\ 0 \le \theta \le \dfrac{\pi}{4}$ (since $0 < x < 1$)

$1 + x^2 = 1 + \tan^2\theta = \sec^2\theta$

When $x = 0$, $\theta = 0$ and when $x = 1$, $\theta = \dfrac{\pi}{4}$.

$$\begin{aligned} \int_0^1 \frac{dx}{(1+x^2)^2} &= \int_0^{\pi/4} \frac{\sec^2\theta}{\sec^4\theta}\,d\theta \\ &= \int_0^{\pi/4} \cos^2\theta\,d\theta \\ &= \int_0^{\pi/4} \frac{1+\cos 2\theta}{2}\,d\theta \\ &= \left[\frac{1}{2}\theta + \frac{\sin 2\theta}{4}\right]_0^{\pi/4} \\ &= \frac{\pi}{8} + \frac{1}{4} \end{aligned}$$

$\text{Volume} = 4\pi\left(\dfrac{\pi}{8} + \dfrac{1}{4}\right) = \pi\left(\dfrac{\pi}{2} + 1\right) \approx 8.076$

47. (a) $\dfrac{dx}{x(1000 - x)} = \dfrac{1}{250}\,dt$

$\dfrac{1}{x(1000-x)} = \dfrac{A}{x} + \dfrac{B}{1000-x}$

$$\begin{aligned} 1 &= A(1000 - x) + Bx \\ &= (-A + B)x + 1000A \end{aligned}$$

Equating the coefficients and solving for A and B gives

$A = \dfrac{1}{1000},\ B = \dfrac{1}{1000}$

$$\begin{aligned} \int \frac{dx}{x(1000-x)} &= \int \frac{(1/1000)\,dx}{x} + \int \frac{(1/1000)\,dx}{1000-x} \\ &= \frac{1}{1000}\ln x - \frac{1}{1000}\ln(1000 - x) + C_1 \\ &= \frac{1}{1000}\ln\frac{x}{1000-x} + C_1 \end{aligned}$$

$$\frac{1}{1000}\ln\frac{x}{1000-x} = \frac{t}{250} + C_2$$

$$\ln\frac{x}{1000-x} = 4t + C$$

$$\frac{x}{1000-x} = e^{4t+C} = Ae^{4t}$$

47. continued

When $t = 0$, $x = 2$.

$$\frac{2}{998} = A \text{ or } A = \frac{1}{499}$$

$$\frac{x}{1000 - x} = \frac{1}{499}e^{4t}$$

$$x = \frac{1000}{499}e^{4t} - \frac{x}{499}e^{4t}$$

$$x\left(1 + \frac{e^{4t}}{499}\right) = \frac{1000e^{4t}}{499}$$

$$x = \frac{1000e^{4t}}{499 + e^{4t}}$$

$$\text{or } x = \frac{1000}{1 + 499e^{-4t}}$$

(b) $500 = \dfrac{1000}{1 + 499e^{-4t}}$

$$1 + 499e^{-4t} = 2$$

$$e^{-4t} = \frac{1}{499}$$

$$t = -\frac{1}{4}\ln\frac{1}{499} \approx 1.553$$

Half the population will have heard the rumor in about 1.553 days.

(c) $\dfrac{dx}{dt} = \dfrac{1}{250}x(1000 - x)$

$\dfrac{dx}{dt}$ will be greatest when $y = x(1000 - x)$ is greatest.

This occurs when $x = 500$ which occurs when

$t \approx 1.553$ as shown in part (b).

48. $\dfrac{dy}{dx} = -\dfrac{2x}{1 - x^2}$

$$ds = \sqrt{1 + \left(\frac{-2x}{1 - x^2}\right)^2}\, dx$$

$$= \sqrt{\frac{x^4+2x^2+1}{(1 - x^2)^2}}$$

$$= \left|\frac{x^2 + 1}{1 - x^2}\right| dx$$

$$= -\frac{x^2 + 1}{x^2 - 1}\, dx \text{ for } 0 \le x \le \frac{1}{2}$$

$$\text{Arc length} = \int_0^{1/2} \left(-\frac{x^2 + 1}{x^2 - 1}\right) dx$$

$$\begin{array}{r} 1 \\ x^2 - 1\,\overline{)\,x^2 + 1} \\ \underline{x^2 - 1} \\ 2 \end{array}$$

$$-\frac{x^2 + 1}{x^2 - 1} = -1 - \frac{2}{x^2 - 1}$$

$$\frac{2}{x^2 - 1} = \frac{A}{x - 1} + \frac{B}{x + 1}$$

where $A = 1$ and $B = -1$

$$\int_0^{1/2} \left(-\frac{x^2 + 1}{x^2 - 1}\right) dx = \int_0^{1/2} \left(-1 - \frac{1}{x - 1} + \frac{1}{x + 1}\right) dx$$

$$= \Big[-x - \ln|x - 1| + \ln|x + 1|\Big]_0^{1/2}$$

$$= -\frac{1}{2} - \ln\frac{1}{2} + \ln\frac{3}{2}$$

$$= \ln 3 - \frac{1}{2}$$

49. (a) From the figure, $\tan\dfrac{x}{2} = \dfrac{\sin x}{1 + \cos x}$.

(b) From part (a), $z = \dfrac{\sin x}{1 + \cos x}$

$$z(1 + \cos x) = \sin x$$

$$z^2(1 + \cos x)^2 = \sin^2 x$$

$$z^2(1 + \cos x)^2 = 1 - \cos^2 x$$

$$z^2(1 + \cos x)^2 - (1 - \cos x)(1 + \cos x) = 0$$

$$(1 + \cos x)(z^2 + z^2\cos x - 1 + \cos x) = 0$$

$$1 + \cos x = 0 \quad \text{or} \quad (z^2 + 1)\cos x = 1 - z^2$$

$$\cos x = -1 \qquad \cos x = \frac{1 - z^2}{1 + z^2}$$

$\cos x = -1$ does not make sense in this case.

(c) From part (b), $\cos x = \dfrac{1 - z^2}{1 + z^2}$

$$\sin^2 x = 1 - \cos^2 x$$

$$= 1 - \frac{(1 - z^2)^2}{(1 + z^2)^2}$$

$$= \frac{(1 + z^2)^2 - (1 - z^2)^2}{(1 + z^2)^2}$$

$$= \frac{1 + 2z^2 + z^4 - 1 + 2z^2 - z^4}{(1 + z^2)^2}$$

$$= \frac{4z^2}{(1 + z^2)^2}$$

$$\sin x = \pm\frac{2z}{1 + z^2}$$

Only $\sin x = \dfrac{2z}{1 + z^2}$ makes sense in this case.

(d) $z = \tan\dfrac{x}{2}$

$$dz = \left(\frac{1}{2}\sec^2\frac{x}{2}\right) dx$$

$$dz = \frac{1}{2}\left(1 + \tan^2\frac{x}{2}\right) dx$$

$$dz = \frac{1}{2}(1 + z^2)\, dx$$

$$dx = \frac{2\, dz}{1 + z^2}$$

50. $$\int \frac{dx}{1+\sin x} = \int \frac{\frac{2\,dz}{1+z^2}}{1+\frac{2z}{1+z^2}}$$

$$= \int \frac{2\,dz}{z^2+2z+1}$$

$$= \int \frac{2\,dz}{(z+1)^2} = -\frac{2}{z+1} + C$$

$$= -\frac{2}{\tan\frac{x}{2}+1} + C$$

51. $$\int \frac{dx}{1-\cos x} = \int \frac{\frac{2\,dz}{1+z^2}}{1-\frac{1-z^2}{1+z^2}}$$

$$= \int \frac{dz}{z^2}$$

$$= -\frac{1}{z} + C = -\frac{1}{\tan\frac{x}{2}} + C$$

52. $$\int \frac{d\theta}{1-\sin\theta} = \int \frac{\frac{2\,dz}{1+z^2}}{1-\frac{2z}{1+z^2}}$$

$$= \int \frac{2\,dz}{z^2-2z+1}$$

$$= \int \frac{2\,dz}{(z-1)^2}$$

$$= -\frac{2}{z-1} + C$$

$$= -\frac{2}{\tan\frac{\theta}{2}-1} + C$$

$$= \frac{2}{1-\tan\frac{\theta}{2}} + C$$

53. $$\int \frac{dt}{1+\sin t+\cos t} = \int \frac{\frac{2\,dz}{1+z^2}}{1+\frac{2z}{1+z^2}+\frac{1-z^2}{1+z^2}}$$

$$= \int \frac{dz}{z+1}$$

$$= \ln\left|z+1\right| + C$$

$$= \ln\left|\tan\frac{t}{2}+1\right| + C$$

■ Chapter 8 Review Exercises

(pp. 454–455)

1. $\displaystyle \lim_{t\to 0} \frac{t-\ln(1+2t)}{t^2} = \lim_{t\to 0} \frac{1-\frac{2}{1+2t}}{2t} = \infty$ for $t\to 0^-$ and $-\infty$ for $t\to 0^+$

The limit does not exist.

2. $$\lim_{t\to 0} \frac{\tan 3t}{\tan 5t} = \lim_{t\to 0} \frac{3\sec^2 3t}{5\sec^2 5t} = \frac{3}{5}$$

3. $$\lim_{x\to 0} \frac{x\sin x}{1-\cos x} = \lim_{x\to 0} \frac{x\cos x+\sin x}{\sin x}$$
$$= \lim_{x\to 0} \frac{-x\sin x+\cos x+\cos x}{\cos x} = 2$$

4. The limit leads to the indeterminate form 1^∞.

$$f(x) = x^{1/(1-x)}$$
$$\ln f(x) = \frac{\ln x}{1-x}$$
$$\lim_{x\to 1} \frac{\ln x}{1-x} = \lim_{x\to 1} \frac{1/x}{-1} = -1$$
$$\lim_{x\to 1} x^{1/(1-x)} = \lim_{x\to 1} e^{\ln f(x)} = e^{-1} = \frac{1}{e}$$

5. The limit leads to the indeterminate form ∞^0.

$$f(x) = x^{1/x}$$
$$\ln f(x) = \frac{\ln x}{x}$$
$$\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{1/x}{1} = 0$$
$$\lim_{x\to\infty} x^{1/x} = \lim_{x\to\infty} e^{\ln f(x)} = e^0 = 1$$

6. The limit leads to the indeterminate form 1^∞.

$$f(x) = \left(1+\frac{3}{x}\right)^x$$

$$\ln f(x) = x\ln\left(1+\frac{3}{x}\right) = \frac{\ln\left(1+\frac{3}{x}\right)}{\frac{1}{x}}$$

$$\lim_{x\to\infty} \frac{\ln\left(1+\frac{3}{x}\right)}{\frac{1}{x}} = \lim_{x\to\infty} \frac{\frac{-3/x^2}{1+3/x}}{-\frac{1}{x^2}} = \lim_{x\to\infty} \frac{3x}{x+3} = 3$$

$$\lim_{x\to\infty} \left(1+\frac{3}{x}\right)^x = \lim_{x\to\infty} e^{\ln f(x)} = e^3$$

7. $\displaystyle \lim_{r\to\infty} \frac{\cos r}{\ln r} = 0$ since $|\cos r| \le 1$ and $\ln r\to\infty$ as $r\to\infty$.

8. $$\lim_{\theta\to\pi/2} \left(\theta-\frac{\pi}{2}\right)\sec\theta = \lim_{\theta\to\pi/2} \frac{\theta-\frac{\pi}{2}}{\cos\theta} = \lim_{\theta\to\pi/2} \frac{1}{-\sin\theta} = -1$$

9. $$\lim_{x\to1}\left(\frac{1}{x-1}-\frac{1}{\ln x}\right)=\lim_{x\to1}\left[\frac{\ln x-x+1}{(x-1)\ln x}\right]$$
$$=\lim_{x\to1}\frac{\frac{1}{x}-1}{\frac{x-1}{x}+\ln x}$$
$$=\lim_{x\to1}\frac{1-x}{x-1+x\ln x}$$
$$=\lim_{x\to1}\frac{-1}{1+x/x+\ln x}=-\frac{1}{2}$$

10. The limit leads to the indeterminate form ∞^0.

$$f(x)=\left(1+\frac{1}{x}\right)^x$$

$$\ln f(x)=x\ln\left(1+\frac{1}{x}\right)=\frac{\ln(1+1/x)}{1/x}$$

$$\lim_{x\to0^+}\frac{\ln(1+1/x)}{1/x}=\lim_{x\to0^+}\frac{\frac{-1/x^2}{1+1/x}}{-1/x^2}=\lim_{x\to0^+}\frac{x}{x+1}=0$$

$$\lim_{x\to0^+}\left(1+\frac{1}{x}\right)^x=\lim_{x\to0^+}e^{\ln f(x)}=e^0=1$$

11. The limit leads to the indeterminate form 0^0.

$$f(\theta)=(\tan\theta)^\theta$$

$$\ln f(\theta)=\theta\ln(\tan\theta)=\frac{\ln(\tan\theta)}{1/\theta}$$

$$\lim_{x\to0^+}\frac{\ln(\tan\theta)}{1/\theta}=\lim_{x\to0^+}\frac{\frac{\sec^2\theta}{\tan\theta}}{-\frac{1}{\theta^2}}$$
$$=\lim_{x\to0^+}-\frac{\theta^2}{\sin\theta\cos\theta}$$
$$=\lim_{x\to0^+}\frac{-2\theta}{-\sin^2\theta+\cos^2\theta}=0$$

$$\lim_{x\to0^+}(\tan\theta)^\theta=\lim_{x\to0^+}e^{\ln f(\theta)}=e^0=1$$

12. $$\lim_{\theta\to\infty}\theta^2\sin\left(\frac{1}{\theta}\right)=\lim_{t\to0^+}\frac{\sin t}{t^2}=\lim_{t\to0^+}\frac{\cos t}{2t}=\infty$$

13. $$\lim_{x\to\infty}\frac{x^3-3x^2+1}{2x^2+x-3}=\lim_{x\to\infty}\frac{3x^2-6x}{4x+1}=\lim_{x\to\infty}\frac{6x-6}{4}=\infty$$

14. $$\lim_{x\to\infty}\frac{3x^2-x+1}{x^4-x^3+2}=\lim_{x\to\infty}\frac{6x-1}{4x^3-3x^2}=\lim_{x\to\infty}\frac{6}{12x^2-6x}=0$$

15. $$\lim_{x\to\infty}\frac{f(x)}{g(x)}=\lim_{x\to\infty}\frac{x}{5x}=\frac{1}{5}$$
f grows at the same rate as g.

16. $$\lim_{x\to\infty}\frac{f(x)}{g(x)}=\lim_{x\to\infty}\frac{\log_2 x}{\log_3 x}=\lim_{x\to\infty}\frac{(\ln x)/(\ln 2)}{(\ln x)/(\ln 3)}=\frac{\ln 3}{\ln 2}$$
f grows at the same rate as g.

17. $$\lim_{x\to\infty}\frac{f(x)}{g(x)}=\lim_{x\to\infty}\frac{x}{x+1/x}=\lim_{x\to\infty}\frac{1}{1-1/x^2}=1$$
f grows at the same rate as g.

18. $$\lim_{x\to\infty}\frac{f(x)}{g(x)}=\lim_{x\to\infty}\frac{x/100}{xe^{-x}}=\lim_{x\to\infty}\frac{e^x}{100}=\infty$$
f grows faster than g.

19. $$\lim_{x\to\infty}\frac{f(x)}{g(x)}=\lim_{x\to\infty}\frac{x}{\tan^{-1}x}=\infty \text{ since}$$
$$\lim_{x\to\infty}\tan^{-1}x=\frac{\pi}{2}\text{ and }\lim_{x\to\infty}x=\infty$$
f grows faster than g.

20. $$\lim_{x\to\infty}\frac{f(x)}{g(x)}=\lim_{x\to\infty}\frac{\csc^{-1}x}{1/x}$$
$$=\lim_{x\to\infty}\frac{-\frac{1}{x\sqrt{x^2-1}}}{-\frac{1}{x^2}}$$
$$=\lim_{x\to\infty}\frac{x}{\sqrt{x^2-1}}$$
$$=\lim_{x\to\infty}\sqrt{\frac{x^2}{x^2-1}}$$
$$=\lim_{x\to\infty}\sqrt{\frac{1}{1-1/x^2}}=1$$
f grows at the same rate as g.

21. $$\lim_{x\to\infty}\frac{f(x)}{g(x)}=\lim_{x\to\infty}\frac{x^{\ln x}}{x^{\log_2 x}}$$
$$=\lim_{x\to\infty}x^{\ln x-\log_2 x}$$
$$=\lim_{x\to\infty}x^{\ln x-(\ln x)/\ln 2}$$
$$=\lim_{x\to\infty}x^{(\ln x)(1-1/\ln 2)}$$
$$=\lim_{x\to\infty}\left(\frac{1}{x}\right)^{(\ln x)(1/\ln 2-1)}=0$$
Note that $1-\frac{1}{\ln 2}<0$ since $\ln 2<1$.

f grows slower than g.

22. $$\lim_{x\to\infty}\frac{f(x)}{g(x)}=\lim_{x\to\infty}\frac{3^{-x}}{2^{-x}}=\lim_{x\to\infty}\frac{2^x}{3^x}=\lim_{x\to\infty}\left(\frac{2}{3}\right)^x=0\text{ since }\frac{2}{3}<1.$$
f grows slower than g.

23. $$\lim_{x\to\infty}\frac{f(x)}{g(x)}=\lim_{x\to\infty}\frac{\ln 2x}{\ln x^2}=\lim_{x\to\infty}\frac{\ln x+\ln 2}{2\ln x}=\lim_{x\to\infty}\frac{1/x}{2/x}=\frac{1}{2}$$
f grows at the same rate as g.

24. $$\lim_{x\to\infty}\frac{f(x)}{g(x)}=\lim_{x\to\infty}\frac{10x^3+2x^2}{e^x}$$
$$=\lim_{x\to\infty}\frac{30x^2+4x}{e^x}$$
$$=\lim_{x\to\infty}\frac{60x+4}{e^x}$$
$$=\lim_{x\to\infty}\frac{60}{e^x}=0$$
f grows slower than g.

25. $$\begin{aligned}\lim_{x\to\infty}\frac{f(x)}{g(x)} &= \lim_{x\to\infty}\frac{\tan^{-1}(1/x)}{1/x}\\ &= \lim_{x\to\infty}\frac{\frac{1}{1+(1/x)^2}(-x^{-2})}{-x^{-2}}\\ &= \lim_{x\to\infty}\frac{1}{1+(1/x)^2} = 1\end{aligned}$$

f grows at the same rate as g.

26. $$\begin{aligned}\lim_{x\to\infty}\frac{f(x)}{g(x)} &= \lim_{x\to\infty}\frac{\sin^{-1}(1/x)}{(1/x^2)}\\ &= \lim_{x\to\infty}\frac{\frac{1}{\sqrt{1-(1/x^2)}}(-x^{-2})}{-2x^{-3}}\\ &= \lim_{x\to\infty}\frac{x}{2\sqrt{1-(1/x)^2}} = \infty\end{aligned}$$

f grows faster than g.

27. **(a)** $$\lim_{x\to 0} f(x) = \lim_{x\to 0}\frac{2^{\sin x}-1}{e^x-1} = \lim_{x\to 0}\frac{(\ln 2)(\cos x)2^{\sin x}}{e^x} = \ln 2$$

(b) Define $f(0) = \ln 2$.

28. **(a)** $$\begin{aligned}\lim_{x\to 0^+} f(x) &= \lim_{x\to 0^+} x\ln x\\ &= \lim_{x\to 0^+}\frac{\ln x}{1/x}\\ &= \lim_{x\to 0^+}\frac{1/x}{-1/x^2}\\ &= \lim_{x\to 0^+}(-x) = 0\end{aligned}$$

(b) Define $f(0) = 0$.

29. $$\lim_{x\to\infty}\frac{\frac{1}{x^2}+\frac{1}{x^4}}{\frac{1}{x^2}} = \lim_{x\to\infty}\left(1+\frac{1}{x^2}\right) = 1 \le 1$$

True

30. $$\lim_{x\to\infty}\frac{\frac{1}{x^2}+\frac{1}{x^4}}{\frac{1}{x^4}} = \lim_{x\to\infty}(x^2+1) = \infty$$

False

31. $$\lim_{x\to\infty}\frac{x}{x+\ln x} = \lim_{x\to\infty}\frac{1}{1+\frac{1}{x}} = 1 \ne 0$$

False

32. $$\lim_{x\to\infty}\frac{\ln(\ln x)}{\ln x} = \lim_{x\to\infty}\frac{\frac{1}{x\ln x}}{\frac{1}{x}} = \lim_{x\to\infty}\frac{1}{\ln x} = 0$$

True

33. $$\lim_{x\to\infty}\frac{\tan^{-1}x}{1} = \frac{\pi}{2} \le \frac{\pi}{2}$$

True

34. $$\lim_{x\to\infty}\frac{\frac{1}{x^4}}{\frac{1}{x^2}+\frac{1}{x^4}} = \lim_{x\to\infty}\frac{1}{x^2+1} = 0 \le 1$$

True

35. $$\lim_{x\to\infty}\frac{\frac{1}{x^4}}{\frac{1}{x^2}+\frac{1}{x^4}} = \lim_{x\to\infty}\frac{1}{x^2+1} = 0$$

True

36. $$\lim_{x\to\infty}\frac{\ln x}{x+1} = \lim_{x\to\infty}\frac{\frac{1}{x}}{1} = 0$$

True

37. $$\lim_{x\to\infty}\frac{\ln 2x}{\ln x} = \lim_{x\to\infty}\frac{\frac{1}{x}}{\frac{1}{x}} = 1 \le 1$$

True

38. $$\lim_{x\to\infty}\frac{\sec^{-1}x}{1} = \frac{\pi}{2} \le \frac{\pi}{2}$$

True

39. $x = 3\sin\theta$, $dx = 3\cos\theta\, d\theta$, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

$$\begin{aligned}\int\frac{dx}{\sqrt{9-x^2}} &= \int\frac{3\cos\theta\, d\theta}{|3\cos\theta|}\\ &= \int d\theta\\ &= \theta + C\\ &= \sin^{-1}\frac{x}{3} + C\end{aligned}$$

$$\begin{aligned}\int_0^3\frac{dx}{\sqrt{9-x^2}} &= \lim_{b\to 3^-}\int_0^b\frac{dx}{\sqrt{9-x^2}}\\ &= \lim_{b\to 3^-}\left[\sin^{-1}\frac{x}{3}\right]_0^b\\ &= \lim_{b\to 3^-}\left(\sin^{-1}\frac{b}{3}\right) = \frac{\pi}{2}\end{aligned}$$

40. $u = \ln x \qquad dv = dx$

$du = \dfrac{1}{x}\,dx \qquad v = x$

$$\int \ln x\,dx = x\ln x - \int dx = x\ln x - x + C$$

$$\begin{aligned}\int_0^1 \ln x\,dx &= \lim_{b\to 0^+}\int_b^1 \ln x\,dx\\ &= \lim_{b\to 0^+}\Big[x\ln x - x\Big]_b^1\\ &= \lim_{b\to 0^+}(-1 - b\ln b + b)\\ &= -1 - \lim_{b\to 0^+}\frac{\ln b}{1/b}\\ &= -1 - \lim_{b\to 0^+}\frac{1/b}{-1/b^2}\\ &= -1 - \lim_{b\to 0^+}(-b) = -1\end{aligned}$$

41. $\displaystyle\int_{-1}^1 \frac{dy}{y^{2/3}} = \int_{-1}^0 \frac{dy}{y^{2/3}} + \int_0^1 \frac{dy}{y^{2/3}}$

$$\begin{aligned}\int_{-1}^0 \frac{dy}{y^{2/3}} &= \lim_{b\to 0^-}\int_{-1}^b y^{-2/3}\,dy\\ &= \lim_{b\to 0^-}\Big[3y^{1/3}\Big]_{-1}^b\\ &= \lim_{b\to 0^-}[3b^{1/3} + 3] = 3\end{aligned}$$

$$\begin{aligned}\int_0^1 \frac{dy}{y^{2/3}} &= \lim_{b\to 0^+}\int_b^1 y^{-2/3}\,dy\\ &= \lim_{b\to 0^+}\Big[3y^{1/3}\Big]_b^1\\ &= \lim_{b\to 0^+}[3 - 3b^{1/3}] = 3\end{aligned}$$

$$\int_{-1}^1 \frac{dy}{y^{2/3}} = 3 + 3 = 6$$

42. $\displaystyle\int_{-2}^0 \frac{d\theta}{(\theta+1)^{3/5}} = \int_{-2}^{-1} \frac{d\theta}{(\theta+1)^{3/5}} + \int_{-1}^0 \frac{d\theta}{(\theta+1)^{3/5}}$

$$\begin{aligned}\int_{-2}^{-1} \frac{d\theta}{(\theta+1)^{3/5}} &= \lim_{b\to -1^-}\int_{-2}^b \frac{d\theta}{(\theta+1)^{3/5}}\\ &= \lim_{b\to -1^-}\left[\frac{5}{2}(\theta+1)^{2/5}\right]_{-2}^b\\ &= \lim_{b\to -1^-}\left[\frac{5}{2}(b+1)^{2/5} - \frac{5}{2}\right] = -\frac{5}{2}\end{aligned}$$

$$\begin{aligned}\int_{-1}^0 \frac{d\theta}{(\theta+1)^{3/5}} &= \lim_{b\to -1^+}\int_b^0 \frac{d\theta}{(\theta+1)^{3/5}}\\ &= \lim_{b\to -1^+}\left[\frac{5}{2}(\theta+1)^{2/5}\right]_b^0\\ &= \lim_{b\to -1^+}\left[\frac{5}{2} - \frac{5}{2}(b+1)^{2/5}\right] = \frac{5}{2}\end{aligned}$$

$$\int_{-2}^0 \frac{d\theta}{(\theta+1)^{3/5}} = -\frac{5}{2} + \frac{5}{2} = 0$$

43. $\displaystyle\int_3^\infty \frac{2\,dx}{x^2 - 2x} = \lim_{b\to\infty}\int_3^b \frac{2\,dx}{x(x-2)}$

$$\frac{2}{x(x-2)} = \frac{A}{x} + \frac{B}{x-2}$$

$$2 = A(x-2) + Bx = (A+B)x - 2A$$

where $A = -1$, $B = 1$.

$$\begin{aligned}\int_3^\infty \frac{2\,dx}{x(x-2)} &= \lim_{b\to\infty}\int_3^b\left(-\frac{1}{x} + \frac{1}{x-2}\right)dx\\ &= \lim_{b\to\infty}\Big[-\ln|x| + \ln|x+2|\Big]_3^b\\ &= \lim_{b\to\infty}\left[\ln\frac{x-2}{x}\right]_3^b\\ &= \lim_{b\to\infty}\left(\ln\frac{b+2}{b} - \ln\frac{1}{3}\right) = \ln 3\end{aligned}$$

44. $\displaystyle\int_1^\infty \frac{3x-1}{4x^3 - x^2}\,dx = \lim_{b\to\infty}\int_1^b \frac{3x-1}{x^2(4x-1)}\,dx$

$$\frac{3x-1}{x^2(4x-1)} = \frac{A}{4x-1} + \frac{B}{x} + \frac{C}{x^2}$$

$$\begin{aligned}3x - 1 &= Ax^2 + Bx(4x-1) + C(4x-1)\\ &= (A+4B)x^2 + (-B+4C)x - C\end{aligned}$$

where $A = -4$, $B = 1$, $C = 1$

$$\begin{aligned}\int_1^\infty \frac{3x-1}{4x^3 - x^2}\,dx &= \lim_{b\to\infty}\int_1^b\left(-\frac{4}{4x-1} + \frac{1}{x} + \frac{1}{x^2}\right)dx\\ &= \lim_{b\to\infty}\left[-\ln|4x-1| + \ln|x| - \frac{1}{x}\right]_1^b\\ &= \lim_{b\to\infty}\left(\ln\frac{b}{4b-1} - \frac{1}{b} + \ln 3 + 1\right)\\ &= \ln\frac{1}{4} + \ln 3 + 1 = \ln\frac{3}{4} + 1\end{aligned}$$

45. $u = x^2 \qquad dv = e^{-x}\,dx$

$du = 2x\,dx \qquad v = -e^{-x}$

$$\int x^2e^{-x}\,dx = -x^2e^{-x} + \int 2x\,e^{-x}\,dx$$

$u = 2x \qquad dv = e^{-x}\,dx$

$du = 2\,dx \qquad v = -e^{-x}$

$$\begin{aligned}\int x^2e^{-x}\,dx &= -x^2e^{-x} - 2x\,e^{-x} + \int 2e^{-x}\,dx\\ &= -x^2e^{-x} - 2x\,e^{-x} - 2\,e^{-x} + C\end{aligned}$$

$$\begin{aligned}\int_0^\infty x^2e^{-x}\,dx &= \lim_{b\to\infty}\int_0^b x^2e^{-x}\,dx\\ &= \lim_{b\to\infty}\Big[-x^2e^{-x} - 2x\,e^{-x} - 2\,e^{-x}\Big]_0^b\\ &= \lim_{b\to\infty}\left[-\frac{b^2}{e^b} - \frac{2b}{e^b} - \frac{2}{e^b} + 2\right] = 2\end{aligned}$$

46. $u = x \qquad dv = e^{3x}\,dx$

$du = dx \qquad v = \frac{1}{3}e^{3x}$

$$\begin{aligned}\int x\,e^{3x}\,dx &= \frac{1}{3}x\,e^{3x} - \int \frac{1}{3}\,e^{3x}\,dx \\ &= \frac{1}{3}x\,e^{3x} - \frac{1}{9}\,e^{3x} + C\end{aligned}$$

$$\begin{aligned}\int_{-\infty}^{0} x\,e^{3x}\,dx &= \lim_{b\to-\infty} \int_{b}^{0} x\,e^{3x}\,dx \\ &= \lim_{b\to-\infty} \left[\frac{1}{3}x\,e^{3x} - \frac{1}{9}\,e^{3x}\right]_{b}^{0} \\ &= \lim_{b\to-\infty} \left[-\frac{1}{9} - \frac{1}{3}b\,e^{3b} + \frac{1}{9}\,e^{3b}\right] = -\frac{1}{9}\end{aligned}$$

47. $\dfrac{4t^3 + t - 1}{t^2(t-1)(t^2+1)} = \dfrac{A}{t} + \dfrac{B}{t^2} + \dfrac{C}{t-1} + \dfrac{Dt+E}{t^2+1}$

$$\begin{aligned}4t^3 + t - 1 &= At(t-1)(t^2+1) + B(t-1)(t^2+1) \\ &\quad + Ct^2(t^2+1) + (Dt+E)t^2(t-1) \\ &= (A + C + D)t^4 + (-A + B - D + E)t^3 \\ &\quad + (A - B + C - E)t^2 + (-A + B)t - B\end{aligned}$$

Equating coefficients of like terms gives

$A + C + D = 0,\ -A + B - D + E = 4,$

$A - B + C - E = 0,\ -A + B = 1$ and $-B = -1$.

Solving the system simultaneously yields

$A = 0,\ B = 1,\ C = 2,\ D = -2,\ E = 1.$

$$\begin{aligned}&\int \frac{4t^3 + t - 1}{t^2(t-1)(t^2+1)}\,dt \\ &= \int \frac{dt}{t^2} + \int \frac{2\,dt}{t-1} + \int \frac{-2t+1}{t^2+1}\,dt \\ &= \int \frac{dt}{t^2} + \int \frac{2\,dt}{t-1} - \int \frac{2t\,dt}{t^2+1} + \int \frac{1\,dt}{t^2+1} \\ &= -\frac{1}{t} + 2\ln\left|t-1\right| - \ln\left|t^2+1\right| + \tan^{-1} t + C \\ &= -\frac{1}{t} + \ln\frac{(t-1)^2}{t^2+1} + \tan^{-1} t + C\end{aligned}$$

$$\begin{aligned}&\int_{-\infty}^{\infty} \frac{4t^3 + t - 1}{t^2(t-1)(t^2+1)}\,dt \\ &= \int_{-\infty}^{-1} \frac{4t^3 + t - 1}{t^2(t-1)(t^2+1)}\,dt + \int_{-1}^{0} \frac{4t^3 + t - 1}{t^2(t-1)(t^2+1)}\,dt \\ &\quad + \int_{0}^{1/2} \frac{4t^3 + t - 1}{t^2(t-1)(t^2+1)}\,dt + \int_{1/2}^{1} \frac{4t^3 + t - 1}{t^2(t-1)(t^2+1)}\,dt \\ &\quad + \int_{1}^{2} \frac{4t^3 + t - 1}{t^2(t-1)(t^2+1)}\,dt + \int_{2}^{\infty} \frac{4t^3 + t - 1}{t^2(t-1)(t^2+1)}\,dt\end{aligned}$$

Note that the integral must be broken up since the integrand has infinite discontinuities at $t = 0$ and $t = 1$.

$$\begin{aligned}&\int_{-1}^{0} \frac{4t^3 + t - 1}{t^2(t-1)(t^2+1)}\,dt \\ &= \lim_{b\to0^-} \int_{-1}^{b} \frac{4t^3 + t - 1}{t^2(t-1)(t^2+1)}\,dt \\ &= \lim_{b\to0^-} \left[-\frac{1}{t} + \ln\frac{(t-1)^2}{t^2+1} + \tan^{-1} t\right]_{-1}^{b} \\ &= \lim_{b\to0^-} \left[-\frac{1}{b} + \ln\frac{(b-1)^2}{b^2+1} + \tan^{-1}\right. \\ &\left. b - 1 - \ln 2 + \frac{\pi}{4}\right] = \infty\end{aligned}$$

Since this limit diverges, the given integral diverges.

48. $\displaystyle\int_{-\infty}^{\infty} \frac{4\,dx}{x^2+16} = \int_{-\infty}^{0} \frac{4\,dx}{x^2+16} + \int_{0}^{\infty} \frac{4\,dx}{x^2+16}$

$\displaystyle\int \frac{4\,dx}{x^2+16} = \tan^{-1}\frac{x}{4} + C$ using Formula 16 with $a = 4$

$$\begin{aligned}\int_{-\infty}^{0} \frac{4\,dx}{x^2+16} &= \lim_{b\to-\infty} \int_{b}^{0} \frac{4\,dx}{x^2+16} \\ &= \lim_{b\to-\infty} \left[\tan^{-1}\frac{x}{4}\right]_{b}^{0} \\ &= \lim_{b\to-\infty} \left(-\tan^{-1}\frac{b}{4}\right) = \frac{\pi}{2}\end{aligned}$$

$$\begin{aligned}\int_{0}^{\infty} \frac{4\,dx}{x^2+16} &= \lim_{b\to\infty} \int_{0}^{b} \frac{4\,dx}{x^2+16} \\ &= \lim_{b\to\infty} \left[\tan^{-1}\frac{x}{4}\right]_{0}^{b} \\ &= \lim_{b\to\infty} \left(\tan^{-1}\frac{b}{4}\right) = \frac{\pi}{2}\end{aligned}$$

$$\int_{-\infty}^{\infty} \frac{4\,dx}{x^2+16} = \frac{\pi}{2} + \frac{\pi}{2} = \pi$$

49. Use the limit comparison test with $f(\theta) = \dfrac{1}{\sqrt{\theta^2+1}}$ and $g(\theta) = \dfrac{1}{\theta}$. Both are positive continuous functions on $[1, \infty)$.

$$\lim_{\theta\to\infty} \frac{f(\theta)}{g(\theta)} = \lim_{\theta\to\infty} \frac{\sqrt{\theta^2+1}}{\theta} = \lim_{\theta\to\infty} \sqrt{1 + \frac{1}{\theta^2}} = 1$$

Since $\displaystyle\int_{1}^{\infty} g(\theta)\,d\theta = \int_{1}^{\infty} \frac{1}{\theta}\,d\theta$

$$\begin{aligned}&= \lim_{b\to\infty} \int_{1}^{b} \frac{1}{\theta}\,d\theta \\ &= \lim_{b\to\infty} \Big[\ln\theta\Big]_{1}^{b} \\ &= \lim_{b\to\infty} \ln b \\ &= \infty,\end{aligned}$$

we know that $\displaystyle\int_{1}^{\infty} g(\theta)\,d\theta$ diverges and so $\displaystyle\int_{1}^{\infty} f(\theta)\,d\theta$ diverges. This means that the given integral diverges.

50. Evaluate $\int e^{-x}\cos x\,dx$ using integration by parts.

$u = \cos x \qquad dv = e^{-x}\,dx$

$du = -\sin x\,dx \qquad v = -e^{-x}$

$$\int e^{-x}\cos x\,dx = -e^{-x}\cos x - \int \sin x\, e^{-x}\,dx$$

Evaluate $\int \sin x\, e^{-x}\,dx$ using integration by parts.

$u = \sin x \qquad dv = e^{-x}\,dx$

$du = \cos x\,dx \qquad v = -e^{-x}$

$$\int \sin x\, e^{-x}\,dx = -e^{-x}\sin x + \int e^{-x}\cos x\,dx$$

$$\int e^{-x}\cos x\,dx = -e^{-x}\cos x + e^{-x}\sin x - \int e^{-x}\cos x\,dx$$

$$2\int e^{-x}\cos x\,dx = e^{-x}\sin x - e^{-x}\cos x + C_1$$

$$\int e^{-x}\cos x\,dx = \frac{e^{-x}\sin x - e^{-x}\cos x}{2} + C$$

$$\begin{aligned}\int_0^\infty e^{-u}\cos u\,du &= \lim_{b\to\infty}\int_0^b e^{-x}\cos x\,dx\\ &= \lim_{b\to\infty}\left[\frac{e^{-x}\sin x - e^{-x}\cos x}{2}\right]_0^b\\ &= \lim_{b\to\infty}\left[\frac{e^{-b}\sin b - e^{-b}\cos b}{2} + \frac{1}{2}\right]\\ &= \frac{1}{2}\end{aligned}$$

Note that we cannot use a comparison test since $e^{-x}\cos x < 0$ for some values on $[0, \infty)$.

51. $0 \le \frac{1}{z} \le \frac{\ln z}{z}$ on $[e, \infty)$

$$\int_e^\infty \frac{dz}{z} = \lim_{b\to\infty}\int_e^b \frac{dz}{z} = \lim_{b\to\infty}\Big[\ln|z|\Big]_e^b = \lim_{b\to\infty}(\ln b - 1) = \infty$$

Since this integral diverges, $\int_e^\infty \frac{1}{z}\,dz$ diverges, so the given integral diverges.

52. $0 \le \frac{e^{-t}}{\sqrt{t}} \le e^{-t}$ on $[1, \infty)$

$$\begin{aligned}\int_1^\infty e^{-t}\,dt &= \lim_{b\to\infty}\int_1^b e^{-t}\,dt\\ &= \lim_{b\to\infty}\Big[-e^{-t}\Big]_1^b\\ &= \lim_{b\to\infty}\left(-e^{-b} + \frac{1}{e}\right) = \frac{1}{e}\end{aligned}$$

Since this integral converges, the given integral converges.

53. $\int \frac{dx}{e^x + e^{-x}} = \int \frac{dx}{e^{-x}(e^{2x} + 1)} = \int \frac{e^x\,dx}{(e^x)^2 + 1}$

Let $u = e^x$, $du = e^x\,dx$

$$\int \frac{dx}{e^x + e^{-x}} = \int \frac{du}{u^2 + 1} = \tan^{-1} u + C = \tan^{-1} e^x + C$$

$$\int_{-\infty}^\infty \frac{dx}{e^x + e^{-x}} = \int_{-\infty}^0 \frac{dx}{e^x + e^{-x}} + \int_0^\infty \frac{dx}{e^x + e^{-x}}$$

$$\begin{aligned}\int_{-\infty}^0 \frac{dx}{e^x + e^{-x}} &= \lim_{b\to-\infty}\int_b^0 \frac{dx}{e^x + e^{-x}}\\ &= \lim_{b\to-\infty}\Big[\tan^{-1} e^x\Big]_b^0\\ &= \lim_{b\to-\infty}\left(\frac{\pi}{4} - \tan^{-1} e^b\right) = \frac{\pi}{4}\end{aligned}$$

$$\begin{aligned}\int_0^\infty \frac{dx}{e^x + e^{-x}} &= \lim_{b\to\infty}\int_0^b \frac{dx}{e^x + e^{-x}}\\ &= \lim_{b\to\infty}\Big[\tan^{-1} e^x\Big]_0^b\\ &= \lim_{b\to\infty}\left(\tan^{-1} e^b - \frac{\pi}{4}\right)\\ &= \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}\end{aligned}$$

Since these two integrals converge, the given integral converges.

54. The integral has an infinite discontinuity at $x = 0$.

$$\begin{aligned}\int_{-\infty}^\infty \frac{dx}{x^2(1 + e^x)} &= \int_\infty^{-1} \frac{dx}{x^2(1 + e^x)} + \int_{-1}^0 \frac{dx}{x^2(1 + e^x)}\\ &\quad + \int_0^1 \frac{dx}{x^2(1 + e^x)} + \int_1^\infty \frac{dx}{x^2(1 + e^x)}\end{aligned}$$

$0 \le \frac{1}{4x^2} \le \frac{1}{x^2(1 + e^x)}$ on $(0, 1]$ since $1 + e^x \le 4$ on $(0, 1]$.

$$\int_0^1 \frac{dx}{4x^2} = \lim_{b\to0^+}\int_b^1 \frac{dx}{4x^2} = \lim_{b\to0^+}\left[-\frac{1}{4x}\right]_b^1 = \lim_{b\to0^+}\left[-\frac{1}{4} + \frac{1}{4b}\right] = \infty$$

Since this integral diverges, $\int_0^1 \frac{dx}{x^2(1 + e^x)}$ diverges, so the given integral diverges.

55. $x^2 - 7x + 12 = (x - 4)(x - 3)$

$$\frac{2x + 1}{x^2 - 7x + 12} = \frac{A}{x - 4} + \frac{B}{x - 3}$$

$$\begin{aligned}2x + 1 &= A(x - 3) + B(x - 4)\\ &= (A + B)x - 3A - 4B\end{aligned}$$

Equating coefficients of like terms gives

$A + B = 2$ and $-3A - 4B = 1$.

Solving the system simultaneously yields $A = 9$, $B = -7$.

$$\begin{aligned}\int \frac{2x + 1}{x^2 - 7x + 12}\,dx &= \int \frac{9\,dx}{x - 4} + \int \frac{-7\,dx}{x - 3}\\ &= 9\ln|x - 4| - 7\ln|x - 3| + C\end{aligned}$$

56. $\dfrac{8}{x^3(x+2)} = \dfrac{A}{x+2} + \dfrac{B}{x} + \dfrac{C}{x^2} + \dfrac{D}{x^3}$

$$\begin{aligned} 8 &= Ax^3 + Bx^2(x+2) + Cx(x+2) + D(x+2) \\ &= (A+B)x^3 + (2B+C)x^2 + (2C+D)x + 2D \end{aligned}$$

Equating coefficients of like terms gives

$A + B = 0$, $2B + C = 0$, $2C + D = 0$, and $2D = 8$

Solving the system simultaneously yields

$A = -1, B = 1, C = -2, D = 4$

$$\begin{aligned} \int \frac{8\,dx}{x^3(x+2)} &= \int \frac{-dx}{x+2} + \int \frac{dx}{x} + \int \frac{-2\,dx}{x^2} + \int \frac{4\,dx}{x^3} \\ &= -\ln|x+2| + \ln|x| + \frac{2}{x} - \frac{2}{x^2} + C \end{aligned}$$

57. $t^3 + t = t(t^2+1)$

$$\frac{3t^2 + 4t + 4}{t^3 + t} = \frac{A}{t} + \frac{Bt + C}{t^2 + 1}$$

$$\begin{aligned} 3t^2 + 4t + 4 &= A(t^2+1) + (Bt + C)t \\ &= (A+B)t^2 + Ct + A \end{aligned}$$

Equating coefficients of like terms gives

$A + B = 3$, $C = 4$ and $A = 4$.

Solving the system simultaneously yields

$A = 4, B = -1, C = 4$

$$\begin{aligned} \int \frac{3t^2 + 4t + 4}{t^3 + t}\,dt &= \int \frac{4\,dt}{t} + \int \frac{-t+4}{t^2+1}\,dt \\ &= \int \frac{4\,dt}{t} - \int \frac{t\,dt}{t^2+1} + \int \frac{4\,dt}{t^2+1} \\ &= 4\ln|t| - \frac{1}{2}\ln(t^2+1) + 4\tan^{-1} t + C \end{aligned}$$

58. $t^4 + 4t^2 + 3 = (t^2+3)(t^2+1)$

$$\frac{1}{(t^2+1)(t^2+3)} = \frac{At + B}{t^2+1} + \frac{Ct + D}{t^2+3}$$

$$\begin{aligned} 1 &= (At + B)(t^2+3) + (Ct + D)(t^2+1) \\ &= (A+C)t^3 + (B+D)t^2 + (3A+C)t + 3B + D \end{aligned}$$

Equating coefficients of like terms gives

$A + C = 0$, $B + D = 0$, $3A + C = 0$, and $3B + D = 1$

Solving the system simultaneously yields

$A = 0, B = \frac{1}{2}, C = 0, D = -\frac{1}{2}$

$$\begin{aligned} \int \frac{dt}{(t^2+3)(t^2+1)} &= \int \frac{1/2}{t^2+1}\,dt - \int \frac{1/2}{t^2+3}\,dt \\ &= \frac{1}{2}\tan^{-1} t - \frac{1}{2\sqrt{3}}\tan^{-1}\frac{t}{\sqrt{3}} + C \end{aligned}$$

Evaluate the integrals using Formula 16, with $x = t$, $a = 1$ in the first integral and $a = \sqrt{3}$ in the second.

59.
$$\begin{array}{r} 1 \\ x^3 - x\,\overline{)\,x^3 \qquad\quad + 1} \\ \underline{x^3 \qquad\quad - x} \\ x + 1 \end{array}$$

$$\frac{x^3+1}{x^3-x} = 1 + \frac{x+1}{x^3-x} = 1 + \frac{x+1}{x(x^2-1)} = 1 + \frac{x+1}{x(x-1)(x+1)}$$

$$\frac{x+1}{x(x-1)(x+1)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1}$$

$$\begin{aligned} x + 1 &= A(x-1)(x+1) + Bx(x+1) + Cx(x-1) \\ &= (A+B+C)x^2 + (B-C)x - A \end{aligned}$$

Equating coefficients of like terms gives

$A + B + C = 0$, $B - C = 1$, and $-A = 1$.

Solving the system simultaneously yields

$A = -1, B = 1, C = 0$

$$\begin{aligned} \int \frac{x^3+1}{x^3-x}\,dx &= \int dx - \int \frac{dx}{x} + \int \frac{dx}{x-1} \\ &= x - \ln|x| + \ln|x-1| + C \end{aligned}$$

60.
$$\begin{array}{r} x \\ x^2 + 4x + 3\,\overline{)\,x^3 + 4x^2} \\ \underline{x^3 + 4x^2 + 3x} \\ -3x \end{array}$$

$$\frac{x^3 + 4x^2}{x^2 + 4x + 3} = x + \frac{-3x}{x^2 + 4x + 3} = x + \frac{-3x}{(x+1)(x+3)}$$

$$\frac{-3x}{(x+1)(x+3)} = \frac{A}{x+1} + \frac{B}{x+3}$$

$$-3x = A(x+3) + B(x+1)$$

$$-3x = (A+B)x + 3A + B$$

Equating coefficients of like terms gives

$A + B = -3$, $3A + B = 0$

Solving the system simultaneously yields $A = \frac{3}{2}$, $B = -\frac{9}{2}$.

$$\begin{aligned} \int \frac{x^3 + 4x^2}{x^2 + 4x + 3}\,dx &= \int x\,dx + \int \frac{3/2}{x+1}\,dx - \int \frac{9/2}{x+3}\,dx \\ &= \frac{x^2}{2} + \frac{3}{2}\ln|x+1| - \frac{9}{2}\ln|x+3| + C \end{aligned}$$

61. $\dfrac{dy}{y(500-y)} = 0.002\,dx$

$$\frac{1}{y(500-y)} = \frac{A}{y} + \frac{B}{500-y}$$

$$1 = A(500-y) + By$$

$$= (B-A)y + 500A$$

where $A = \dfrac{1}{500}$, $B = \dfrac{1}{500}$.

$$\int \frac{dy}{y(500-y)} = \int \frac{1/500}{y}\,dy + \int \frac{1/500}{500-y}\,dy$$

$$= \frac{1}{500}\ln|y| - \frac{1}{500}\ln|500-y| + C_1$$

$$= \frac{1}{500}\ln\left|\frac{y}{500-y}\right| + C_1$$

$$\frac{1}{500}\ln\left|\frac{y}{500-y}\right| + C_1 = 0.002x + C_2$$

$$\ln\left|\frac{y}{500-y}\right| = x + C$$

$$\frac{y}{500-y} = ke^x$$

Substitute $x = 0$, $y = 20$.

$$\frac{20}{480} = ke^0 \text{ or } k = \frac{1}{24}$$

$$\frac{y}{500-y} = \frac{1}{24}e^x$$

$$24y = 500e^x - ye^x$$

$$(e^x + 24)y = 500e^x$$

$$y = \frac{500\,e^x}{e^x + 24}$$

$$y = \frac{500}{1 + 24\,e^{-x}}$$

62. $\dfrac{dy}{y^2+1} = \dfrac{dx}{x+1}$

$$\int \frac{dy}{y^2+1} = \int \frac{dx}{x+1}$$

$$\tan^{-1} y + C_1 = \ln|x+1| + C_2$$

$$\tan^{-1} y = \ln|x+1| + C$$

Substitute $x = 0$, $y = \dfrac{\pi}{4}$

$$\tan^{-1}\frac{\pi}{4} = C$$

$$\tan^{-1} y = \ln|x+1| + \tan^{-1}\frac{\pi}{4}$$

$$y = \tan\left(\ln|x+1| + \tan^{-1}\frac{\pi}{4}\right)$$

63. $y = \dfrac{1}{3}\tan\theta$, $dy = \dfrac{1}{3}\sec^2\theta\,d\theta$, $-\dfrac{\pi}{2} < \theta < \dfrac{\pi}{2}$

$$1 + 9y^2 = 1 + \tan^2\theta = \sec^2\theta$$

$$\int \frac{3\,dy}{\sqrt{1+9y^2}} = \int \frac{\sec^2\theta\,d\theta}{|\sec\theta|}$$

$$= \int \sec\theta\,d\theta$$

$$= \ln|\sec\theta + \tan\theta| + C$$

$$= \ln\left|\sqrt{1+9y^2} + 3y\right| + C$$

Integrate by using Formula 88 with $a = 1$ and $x = \theta$.

Use Figure 8.18(a) from the text with $a = \dfrac{1}{3}$.

64. $t = \dfrac{1}{3}\sin\theta$, $dt = \dfrac{1}{3}\cos\theta\,d\theta$, $-\dfrac{\pi}{2} \le \theta \le \dfrac{\pi}{2}$

$$1 - 9t^2 = 1 - \sin^2\theta = \cos^2\theta$$

$$\int \sqrt{1-9t^2}\,dt = \int |\cos\theta|\left(\frac{1}{3}\cos\theta\right)d\theta$$

$$= \int \frac{1}{3}\cos^2\theta\,d\theta$$

$$= \int \frac{1+\cos 2\theta}{6}\,d\theta$$

$$= \frac{\theta}{6} + \frac{\sin 2\theta}{12} + C$$

$$= \frac{\theta}{6} + \frac{\sin\theta\cos\theta}{6} + C$$

$$= \frac{\sin^{-1} 3t}{6} + \frac{3t\sqrt{1-9t^2}}{6} + C$$

$$= \frac{1}{6}\sin^{-1} 3t + \frac{1}{2}t\sqrt{1-9t^2} + C$$

Use Figure 8.18(b) with $a = \dfrac{1}{3}$ and $x = t$.

65. $x = \dfrac{3}{5}\sec\theta$, $dx = \dfrac{3}{5}\sec\theta\tan\theta$, $0 \le \theta < \dfrac{\pi}{2}$

$$25x^2 - 9 = 9\sec^2\theta - 9 = 9\tan^2\theta$$

$$\int \frac{5\,dx}{\sqrt{25x^2-9}} = \int \frac{3\sec\theta\tan\theta\,d\theta}{3\,|\tan\theta|}$$

$$= \int \sec\theta\,d\theta$$

$$= \ln|\sec\theta + \tan\theta| + C_1$$

$$= \ln\left|\frac{5x}{3} + \frac{\sqrt{25x^2-9}}{3}\right| + C_1$$

$$= \ln\left(5x + \sqrt{25x^2-9}\right) + C$$

Integrate by using Formula 88 with $a = 1$ and $x = \theta$.

Use Figure 8.18(c) with $a = \dfrac{3}{5}$.

66. $x = \sin\theta,\ dx = \cos\theta\,d\theta,\ -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

$1 - x^2 = \cos^2\theta$

$$\begin{aligned}\int \frac{4x^2\,dx}{(1-x^2)^{3/2}} &= \int \frac{4\sin^2\theta\cos\theta}{|\cos^3\theta|}\,d\theta\\ &= \int \frac{4(1-\cos^2\theta)}{\cos^2\theta}\,d\theta\\ &= \int (4\sec^2\theta - 4)\,d\theta\\ &= 4\tan\theta - 4\theta + C\\ &= \frac{4x}{\sqrt{1-x^2}} - 4\sin^{-1}x + C\end{aligned}$$

Use Figure 8.18(b) with $a = 1$.

67. For $x \ge 0$, $y \ge 0$ on $(0, 1]$.

$$\begin{aligned}\text{Volume} &= \int_0^1 \pi(-\ln x)^2\,dx\\ &= \pi\int_0^1 (\ln x)^2\,dx\\ &= \pi \lim_{b\to 0^+} \int_b^1 (\ln x)^2\,dx\end{aligned}$$

Evaluate $\int (\ln x)^2\,dx$ by using integration by parts.

$u = (\ln x)^2 \qquad dv = dx$

$du = \frac{2\ln x}{x}\,dx \qquad v = x$

$\int (\ln x)^2\,dx = x(\ln x)^2 - \int 2\ln x\,dx$

Evaluate $\int 2\ln x\,dx$ by using integration by parts.

$u = 2\ln x \qquad dv = dx$

$du = \frac{2}{x}\,dx \qquad v = x$

$\int 2\ln x\,dx = 2x\ln x - \int 2\,dx = 2x\ln x - 2x + C$

$\int (\ln x)^2\,dx = x(\ln x)^2 - 2x\ln x + 2x + C$

$$\begin{aligned}\text{Area} &= \pi \lim_{b\to 0^+}\Big[x(\ln x)^2 - 2x\ln x + 2x\Big]_b^1\\ &= \pi \lim_{b\to 0^+}[2 - b(\ln b)^2 + 2b\ln b - 2b]\\ &= 2\pi - \lim_{b\to 0^+}\frac{\pi(\ln b)^2}{1/b} + 2\lim_{b\to 0^+}\frac{\pi\ln b}{1/b}\\ &= 2\pi - \lim_{b\to 0^+}\frac{2\pi(\ln b)(1/b)}{-1/b^2} + 2\lim_{b\to 0^+}\frac{\pi/b}{-1/b^2}\\ &= 2\pi - \lim_{b\to 0^+}\frac{2\pi(\ln b)}{-1/b} + 2\lim_{b\to 0^+}(-\pi b)\\ &= 2\pi - \lim_{b\to 0^+}\frac{2\pi/b}{1/b^2} + 2\pi - \lim_{b\to 0^+}2\pi b = 2\pi\end{aligned}$$

68. For $x \ge 0$, $y \ge 0$ on $[0, \infty)$.

$\text{Area} = \int_0^\infty xe^{-x}\,dx = \lim_{b\to\infty}\int_0^b xe^{-x}\,dx$

Evaluate $\int xe^{-x}\,dx$ by using integration by parts.

$u = x \qquad dv = e^{-x}\,dx$

$du = dx \qquad v = -e^{-x}$

$\int xe^{-x}\,dx = -xe^{-x} + \int e^{-x}\,dx = -xe^{-x} - e^{-x} + C$

$$\begin{aligned}\text{Area} &= \lim_{b\to\infty}\Big[-xe^{-x} - e^{-x}\Big]_0^b\\ &= \lim_{b\to\infty}[-be^{-b} - e^{-b} + 1]\\ &= -\lim_{b\to\infty}\frac{b}{e^b} + 1\\ &= -\lim_{b\to\infty}\frac{1}{e^b} + 1 = 1\end{aligned}$$

69. (a) $\frac{dx}{dt} = k(a-x)^2$

$\frac{dx}{(a-x)^2} = k\,dt$

$\int \frac{dx}{(a-x)^2} = \int k\,dt = kt + C_1$

$\frac{1}{a-x} + C_2 = kt + C_1$

$\frac{1}{a-x} = kt + C$

Substitute $x = 0$, $t = 0$

$\frac{1}{a} = C$

$\frac{1}{a-x} = kt + \frac{1}{a}$

$\frac{1}{kt + 1/a} = a - x$

$x = a - \frac{1}{kt + 1/a}$

69. continued

(b) $\dfrac{dx}{(a-x)(b-x)} = k\,dt$

$$\int \frac{dx}{(a-x)(b-x)} = \int k\,dt = kt + C_1$$

$$\frac{1}{(a-x)(b-x)} = \frac{A}{a-x} + \frac{B}{b-x}$$

$$1 = A(b-x) + B(a-x)$$

$$= (-A-B)x + bA + aB$$

Equating coefficients of like terms gives

$-A - B = 0$ and $bA + aB = 1$

Solving the system simultaneously yields

$$A = -\frac{1}{a-b},\ B = \frac{1}{a-b}$$

$$\int \frac{dx}{(a-x)(b-x)} = \int \frac{-1/(a-b)}{a-x}\,dx + \int \frac{1/(a-b)}{b-x}\,dx$$

$$= \frac{\ln|a-x|}{a-b} - \frac{\ln|b-x|}{a-b} + C_2$$

$$= \frac{1}{a-b}\ln\left|\frac{a-x}{b-x}\right| + C_2$$

$$\frac{1}{a-b}\ln\left|\frac{a-x}{b-x}\right| + C_2 = kt + C_1$$

$$\ln\left|\frac{a-x}{b-x}\right| = (a-b)kt + C$$

$$\frac{a-x}{b-x} = De^{(a-b)kt}$$

Substitute $t = 0$, $x = 0$.

$$\frac{a}{b} = D$$

$$\frac{a-x}{b-x} = \frac{a}{b}e^{(a-b)kt}$$

$$ab - bx = abe^{(a-b)kt} - axe^{(a-b)kt}$$

$$x(ae^{(a-b)kt} - b) = ab(e^{(a-b)kt} - 1)$$

$$x = \frac{ab(e^{(a-b)kt} - 1)}{ae^{(a-b)kt} - b}$$

Multiply the rational expression by $\dfrac{e^{bkt}}{e^{bkt}}$.

$$x = \frac{ab(e^{akt} - e^{bkt})}{ae^{akt} - be^{bkt}}$$

Chapter 9
Infinite Series

■ Section 9.1 Power Series (pp. 457–468)

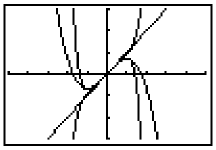
Exploration 1 Power Series for Other Functions

1. $1 - x + x^2 - x^3 + \cdots + (-x)^n + \cdots$.
2. $x - x^2 + x^3 - x^4 + \cdots + (-1)^n x^{n+1} + \cdots$.
3. $1 + 2x + 4x^2 + 8x^3 + \cdots + (2x)^n + \cdots$.
4. $1 - (x-1) + (x-1)^2 - (x-1)^3 + \cdots + (-1)^n(x-1)^n + \cdots$.
5. $\frac{1}{3} - \frac{1}{3}(x-1) + \frac{1}{3}(x-1)^2 - \frac{1}{3}(x-1)^3 + \cdots + \left(-\frac{1}{3}\right)^n(x-1)^n + \cdots$.

   This geometric series converges for $-1 < x - 1 < 1$, which is equivalent to $0 < x < 2$. The interval of convergence is (0, 2).

Exploration 2 A Power Series for $\tan^{-1} x$

1. $1 - x^2 + x^4 - x^6 + \cdots + (-1)^n x^{2n} + \cdots$.
2. $$\tan^{-1} x = \int_0^x \frac{1}{1+t^2}\,dt$$
   $$= \int_0^x (1 - t^2 + t^4 - t^6 + \cdots + (-1)^n t^{2n} + \cdots)\,dt$$
   $$= \left[t - \frac{t^3}{3} + \frac{t^5}{5} - \frac{t^7}{7} + \cdots + (-1)^n\frac{t^{2n+1}}{2n+1} + \cdots\right]_0^x$$
   $$= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots + (-1)^n\frac{x^{2n+1}}{2n+1} \cdots.$$
3. The graphs of the first four partial sums appear to be converging on the interval $(-1, 1)$.

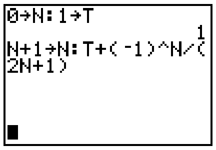


[−5, 5] by [−3, 3]

4. When $x = 1$, the series becomes

   $$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots + \frac{(-1)^n}{2n+1} + \cdots.$$

   This series does appear to converge. The terms are getting smaller, and because they alternate in sign they cause the partial sums to oscillate above and below a limit. The two calculator statements shown below will cause the successive partial sums to appear on the calculator each time the ENTER button is pushed. The partial sums will appear to be approaching a limit of $\pi/4$ (which is $\tan^{-1}(1)$), although very slowly.



```
0→N:1→T
                1
N+1→N:T+(-1)^N/(
2N+1)

■
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