56. Method 1–Compare graph of $y_1 = x^2 \ln x$ with $y_2 = \text{NDER}\left(\frac{x^3 \ln x}{3} - \frac{x^3}{9}\right)$. The graphs should be the same.
Method 2–Compare graph of $y_1 = \text{NINT}(x^2 \ln x)$ with $y_2 = \frac{x^3 \ln x}{3} - \frac{x^3}{9}$. The graphs should be the same or differ only by a vertical translation.

57. **(a)**

$$20{,}000 = 10{,}000(1.063)^t$$
$$2 = 1.063^t$$
$$\ln 2 = t \ln 1.063$$
$$t = \frac{\ln 2}{\ln 1.063} \approx 11.345$$

It will take about 11.3 years.

(b)

$$20{,}000 = 10{,}000e^{0.063t}$$
$$2 = e^{0.063t}$$
$$\ln 2 = 0.063t$$
$$t = \frac{\ln 2}{0.063} \approx 11.002$$

It will take about 11.0 years.

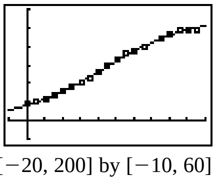
58. **(a)** $f'(x) = \frac{d}{dx}\int_0^x u(t)\,dt = u(x)$

$g'(x) = \frac{d}{dx}\int_3^x u(t)\,dt = u(x)$

(b)

$$C = f(x) - g(x)$$
$$= \int_0^x u(t)\,dt - \int_3^x u(t)\,dt$$
$$= \int_0^x u(t)\,dt + \int_x^3 u(t)\,dt$$
$$= \int_0^3 u(t)\,dt$$

59. **(a)** $y = \frac{56.0716}{1 + 5.894e^{-0.0205x}}$

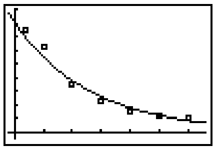


[−20, 200] by [−10, 60]

(b) The carrying capacity is about 56.0716 million people.

(c) Use NDER twice to solve $y'' = 0$. The solution is $x \approx 86.52$, representing (approximately) the year 1887. The population at this time was approximately $P(86.52) \approx 28.0$ million people.

60. **(a)** $T = 79.961(0.9273)^t$



[−1, 33] by [−5, 90]

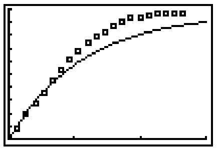
(b) Solving $T(t) = 40$ graphically, we obtain $t \approx 9.2$ sec. The temperature will reach 40° after about 9.2 seconds.

(c) When the probe was removed, the temperature was about $T(0) \approx 79.96$°C.

61.

$$\frac{v_0 m}{k} = \text{coasting distance}$$
$$\frac{(0.86)(30.84)}{k} = 0.97$$
$$k \approx 27.343$$
$$s(t) = \frac{v_0 m}{k}(1 - e^{-(k/m)t})$$
$$s(t) = 0.97(1 - e^{-(27.343/30.84)t})$$
$$s(t) = 0.97(1 - e^{-0.8866t})$$

A graph of the model is shown superimposed on a graph of the data.



[0, 3] by [0,1]

Chapter 7
Applications of Definite Integrals

■ Section 7.1 Integral as Net Change (pp. 363–374)

Exploration 1 Revisiting Example 2

1. $s(t) = \int\left(t^2 - \frac{8}{(t+1)^2}\right)dt = \frac{t^3}{3} + \frac{8}{t+1} + C$

$s(0) = \frac{0^3}{3} + \frac{8}{0+1} + C = 9 \Rightarrow C = 1$

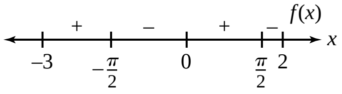
Thus, $s(t) = \frac{t^3}{3} + \frac{8}{t+1} + 1$.

2. $s(1) = \frac{1^3}{3} + \frac{8}{1+1} + 1 = \frac{16}{3}$. This is the same as the answer we found in Example 2a.

3. $s(5) = \frac{5^3}{3} + \frac{8}{5+1} + 1 = 44$. This is the same answer we found in Example 2b.

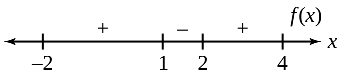
Quick Review 7.1

1. On the interval, $\sin 2x = 0$ when $x = -\frac{\pi}{2}, 0$, or $\frac{\pi}{2}$. Test one point on each subinterval: for $x = -\frac{3\pi}{4}$, $\sin 2x = 1$; for $x = -\frac{\pi}{4}$, $\sin 2x = -1$; for $x = \frac{\pi}{4}$, $\sin 2x = 1$; and for $x = \frac{3\pi}{4}$, $\sin 2x = -1$. The function changes sign at $-\frac{\pi}{2}$, 0, and $\frac{\pi}{2}$. The graph is

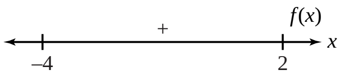


2. $x^2 - 3x + 2 = (x-1)(x-2) = 0$ when $x = 1$ or 2. Test one point on each subinterval: for $x = 0$, $x^2 - 3x + 2 = 2$; for $x = \frac{3}{2}$, $x^2 - 3x + 2 = -\frac{1}{4}$; and for $x = 3$, $x^2 - 3x + 2 = 2$. The function changes sign at 1 and 2.

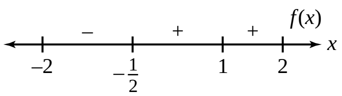
The graph is



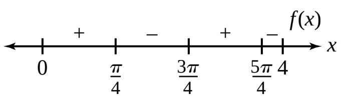
3. $x^2 - 2x + 3 = 0$ has no real solutions, since $b^2 - 4ac = (-2)^2 - 4(1)(3) = -8 < 0$. The function is always positive. The graph is



4. $2x^3 - 3x^2 + 1 = (x-1)^2(2x+1) = 0$ when $x = -\frac{1}{2}$ or 1. Test one point on each subinterval: for $x = -1$, $2x^3 - 3x^2 + 1 = -4$; for $x = 0$, $2x^3 - 3x^2 + 1 = 1$; and $x = \frac{3}{2}$, $2x^3 - 3x^2 + 1 = 1$. The function changes sign at $-\frac{1}{2}$. The graph is

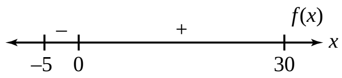


5. On the interval, $x \cos 2x = 0$ when $x = 0, \frac{\pi}{4}, \frac{3\pi}{4}$, or $\frac{5\pi}{4}$. Test one point on each subinterval: for $x = \frac{\pi}{8}$, $x \cos 2x = \frac{\pi\sqrt{2}}{16}$; for $x = \frac{\pi}{2}$, $x \cos 2x = -\frac{\pi}{2}$; for $x = \pi$, $x \cos 2x = \pi$; and for $x = 4$, $x \cos 2x \approx -0.58$. The function changes sign at $\frac{\pi}{4}$, $\frac{3\pi}{4}$, and $\frac{5\pi}{4}$. The graph is

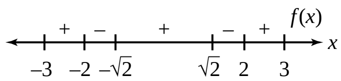


6. $xe^{-x} = 0$ when $x = 0$. On the rest of the interval, xe^{-x} is always positive.

7. $\frac{x}{x^2+1} = 0$ when $x = 0$. Test one point on each subinterval: for $x = -1$, $\frac{x}{x^2+1} = -\frac{1}{2}$; for $x = 1$, $\frac{x}{x^2+1} = \frac{1}{2}$. The function changes sign at 0. The graph is

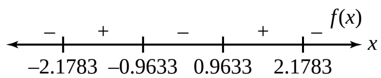


8. $\frac{x^2-2}{x^2-4} = 0$ when $x = \pm\sqrt{2}$ and is undefined when $x = \pm 2$. Test one point on each subinterval: for $x = -\frac{5}{2}$, $\frac{x^2-2}{x^2-4} = \frac{17}{9}$; for $x = -1.9$, $\frac{x^2-2}{x^2-4} \approx -4.13$; for $x = 0$, $\frac{x^2-2}{x^2-4} = \frac{1}{2}$; for $x = 1.9$, $\frac{x^2-2}{x^2-4} \approx -4.13$; and for $x = \frac{5}{2}$, $\frac{x^2-2}{x^2-4} = \frac{17}{9}$. The function changes sign at $-2, -\sqrt{2}, \sqrt{2}$ and 2. The graph is

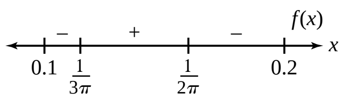


9. $\sec(1 + \sqrt{1 - \sin^2 x}) = \frac{1}{\cos(1 + |\cos x|)}$ is undefined when $x \approx 0.9633 + k\pi$ or $2.1783 + k\pi$ for any integer k. Test for $x = 0$: $\sec(1 + \sqrt{1 - \sin^2 0}) \approx -2.4030$.

Test for $x = \pm 1$: $\sec(1 + \sqrt{1 - \sin^2 1}) \approx 32.7984$. The sign alternates over successive subintervals. The function changes sign at $0.9633 + k\pi$ or $2.1783 + k\pi$, where k is an integer. The graph is



10. On the interval, $\sin\left(\frac{1}{x}\right) = 0$ when $x = \frac{1}{3\pi}$ or $\frac{1}{2\pi}$. Test one point on each subinterval: for $x = 0.1$, $\sin\left(\frac{1}{x}\right) \approx -0.54$; for $x = 0.15$, $\sin\left(\frac{1}{x}\right) \approx 0.37$; and for $x = 0.2$, $\sin\left(\frac{1}{x}\right) \approx -0.96$. The graph changes sign at $\frac{1}{3\pi}$, and $\frac{1}{2\pi}$. The graph is



Section 7.1 Exercises

1. (a) Right when $v(t) > 0$, which is when $\cos t > 0$, i.e., when $0 \le t < \frac{\pi}{2}$ or $\frac{3\pi}{2} < t \le 2\pi$. Left when $\cos t < 0$, i.e., when $\frac{\pi}{2} < t < \frac{3\pi}{2}$. Stopped when $\cos t = 0$, i.e., when $t = \frac{\pi}{2}$ or $\frac{3\pi}{2}$.

(b) Displacement =
$$\int_0^{2\pi} 5\cos t\,dt = 5\Big[\sin t\Big]_0^{2\pi} = 5[\sin 2\pi - \sin 0] = 0$$

(c) Distance $= \int_0^{2\pi} |5\cos t|\,dt$
$$= \int_0^{\pi/2} 5\cos t\,dt + \int_{\pi/2}^{3\pi/2} -5\cos t\,dt + \int_{3\pi/2}^{2\pi} 5\cos t\,dt$$
$$= 5 + 10 + 5 = 20$$

2. (a) Right when $v(t) > 0$, which is when $\sin 3t > 0$, i.e., when $0 < t < \frac{\pi}{3}$. Left when $\sin 3t < 0$, i.e., when $\frac{\pi}{3} < t \le \frac{\pi}{2}$. Stopped when $\sin 3t = 0$, i.e., when $t = 0$ or $\frac{\pi}{3}$.

(b) Displacement $= \int_0^{\pi/2} 6\sin 3t\,dt = 6\Big[-\frac{1}{3}\cos 3t\Big]_0^{\pi/2}$
$$= -2\Big[\cos\frac{3\pi}{2} - \cos 0\Big] = 2$$

(c) Distance $= \int_0^{\pi/2} |6\sin 3t|\,dt$
$$= \int_0^{\pi/3} 6\sin 3t\,dt + \int_{\pi/3}^{\pi/2} -6\sin 3t\,dt = 4 + 2 = 6$$

3. (a) Right when $v(t) = 49 - 9.8t > 0$, i.e., when $0 \le t < 5$.

Left when $49 - 9.8\text{t} < 0$, i.e., when $5 < t \le 10$.

Stopped when $49 - 9.8t = 0$, i.e., when $t = 5$.

(b) Displacement $= \int_0^{10} (49 - 9.8t)\,dt$
$$= \Big[49t - 4.9t^2\Big]_0^{10} = 49[(10 - 10) - 0] = 0$$

(c) Distance $= \int_0^{10} |49 - 9.8t|\,dt$
$$= \int_0^5 (49 - 9.8t)\,dt + \int_5^{10} (-49 + 9.8t)\,dt$$
$$= 122.5 + 122.5 = 245$$

4. (a) Right when
$$v(t) = 6t^2 - 18t + 12 = 6(t-1)(t-2) > 0,$$
i.e., when $0 \le t < 1$. Left when $6(t-1)(t-2) < 0$, i.e., when $1 < t < 2$. Stopped when $6(t-1)(t-2) = 0$, i.e., when $x = 1$, or 2.

(b) Displacement $= \int_0^2 (6t^2 - 18t + 12)\,dt$
$$= \Big[2t^3 - 9t^2 + 12t\Big]_0^2 = [(16 - 36 + 24) - 0] = 4$$

(c) Distance $= \int_0^2 |6t^2 - 18t + 12|\,dt$
$$= \int_0^1 (6t^2 - 18t + 12)\,dt + \int_1^2 (-6t^2 + 18t - 12)\,dt$$
$$= 5 + 1 = 6$$

5. (a) Right when $v(t) > 0$, which is when $\sin t \ne 0$ and $\cos t > 0$, i.e., when $0 < t < \frac{\pi}{2}$ or $\frac{3\pi}{2} < t < 2\pi$. Left when $\sin t \ne 0$ and $\cos t < 0$, i.e., when $\frac{\pi}{2} < t < \pi$ or $\pi < t < \frac{3\pi}{2}$. Stopped when $\sin t = 0$ or $\cos t = 0$, i.e., when $t = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$, or 2π.

(b) Displacement $= \int_0^{2\pi} 5\sin^2 t\cos t\,dt = 5\Big[\frac{1}{3}\sin^3 t\Big]_0^{2\pi}$
$$= 5[0 - 0] = 0$$

(c) Distance $= \int_0^{2\pi} |5\sin^2 t\cos t|\,dt$
$$= \int_0^{\pi/2} 5\sin^2 t\cos t\,dt + \int_{\pi/2}^{3\pi/2} -5\sin^2 t\cos t\,dt + \int_{3\pi/2}^{2\pi} 5\sin^2 t\cos t\,dt$$
$$= \frac{5}{3} + \frac{10}{3} + \frac{5}{3} = \frac{20}{3}$$

6. (a) Right when $v(t) > 0$, which is when $4 - t > 0$, i.e., when $0 \le t < 4$. Left: never, since $\sqrt{4 - t}$ cannot be negative. Stopped when $4 - t = 0$, i.e., when $t = 4$.

(b) Displacement $= \int_0^4 \sqrt{4 - t}\,dt = \Big[-\frac{2}{3}(4 - t)^{3/2}\Big]_0^4$
$$= -\frac{2}{3}[0 - 8] = \frac{16}{3}$$

(c) Distance $= \int_0^4 \sqrt{4 - t}\,dt = \frac{16}{3}$

7. **(a)** Right when $v(t) > 0$, which is when $\cos t > 0$, i.e., when $0 \le t < \frac{\pi}{2}$ or $\frac{3\pi}{2} < t \le 2\pi$. Left when $\cos t < 0$, i.e., when $\frac{\pi}{2} < t < \frac{3\pi}{2}$. Stopped when $\cos t = 0$, i.e., when $t = \frac{\pi}{2}$ or $\frac{3\pi}{2}$.

(b) Displacement $= \int_0^{2\pi} e^{\sin t} \cos t \, dt = \left[e^{\sin t}\right]_0^{2\pi}$
$= [e^0 - e^0] = 0$

(c) Distance $= \int_0^{2\pi} \left|e^{\sin t} \cos t\right| dt = \int_0^{\pi/2} e^{\sin t} \cos t \, dt +$
$\int_{\pi/2}^{3\pi/2} -e^{\sin t} \cos t \, dt + \int_{3\pi/2}^{2\pi} e^{\sin t} \cos t \, dt$
$= (e - 1) + \left(e - \frac{1}{e}\right) + \left(1 - \frac{1}{e}\right) = 2e - \frac{2}{e} \approx 4.7$

8. **(a)** Right when $v(t) > 0$, which is when $0 < t \le 3$. Left: never, since $v(t)$ is never negative. Stopped when $t = 0$.

(b) Displacement $= \int_0^3 \frac{t}{1 + t^2} \, dt = \left[\frac{1}{2} \ln (1 + t^2)\right]_0^3$
$= \frac{1}{2}[\ln (10) - \ln (1)] = \frac{\ln 10}{2} \approx 1.15$

(c) Distance $= \int_0^3 \frac{t}{1 + t^2} \, dt = \frac{\ln 10}{2} \approx 1.15$

9. **(a)** $v(t) = \int a(t) \, dt = t + 2t^{3/2} + C$, and since $v(0) = 0$, $v(t) = t + 2t^{3/2}$. Then $v(9) = 9 + 2(27) = 63$ mph.

(b) First convert units:
$t + 2t^{3/2}$ mph $= \frac{t}{3600} + \frac{t^{3/2}}{1800}$ mi/sec. Then
Distance $= \int_0^9 \left(\frac{t}{3600} + \frac{t^{3/2}}{1800}\right) dt$
$= \left[\frac{t^2}{7200} + \frac{t^{5/2}}{4500}\right]_0^9 = \left[\left(\frac{9}{800} + \frac{27}{500}\right) - 0\right] = 0.06525$ mi
$= 344.52$ ft.

10. **(a)** Displacement $= \int_0^4 (t - 2) \sin t \, dt$
$= \left[\sin t - t \cos t + 2 \cos t\right]_0^4$
$= [(\sin 4 - 4 \cos 4 + 2 \cos 4) - 2] \approx -1.44952$ m

(b) Because the velocity is negative for $0 < t < 2$, positive for $2 < t < \pi$, and negative for $\pi < t \le 4$,
Distance $= \int_0^2 -(t - 2) \sin t \, dt + \int_2^{\pi} (t - 2) \sin t \, dt$
$+ \int_{\pi}^4 -(t - 2) \sin t \, dt$
$= [(2 - \sin 2) + (\pi - \sin 2 - 2)$
$+ (\pi + 2 \cos 4 - \sin 4 - 2)]$
$= 2\pi + 2 \cos 4 - 2 \sin 2 - \sin 4 - 2 \approx 1.91411$ m.

11. **(a)** $v(t) = \int a(t) \, dt = \int -32 \, dt = -32t + C_1$, where $C_1 = v(0) = 90$.
Then $v(3) = -32(3) + 90 = -6$ ft/sec.

(b) $s(t) = \int v(t) \, dt = -16t^2 + 90t + C_2$, where $C_2 = s(0) = 0$. Solve $s(t) = 0$:
$-16t^2 + 90t = 2t(-8t + 45) = 0$
when $t = 0$ or $t = \frac{45}{8} = 5.625$ sec.
The projectile hits the ground at 5.625 sec.

(c) Since starting height = ending height,
Displacement = 0.

(d) Max. Height $= s\left(\frac{5.625}{2}\right)$
$= -16\left(\frac{5.625}{2}\right)^2 + 90\left(\frac{5.625}{2}\right) = 126.5625$, and
Distance = 2(Max. Height) = 253.125 ft.

12. Displacement $= \int_0^c v(t) \, dt = -4 + 5 - 24 = -23$ cm

13. Total distance $= \int_0^c |v(t)| \, dt = 4 + 5 + 24 = 33$ cm

14. At $t = a$, $s = s(0) + \int_0^a v(t) \, dt = 15 - 4 = 11$.
At $t = b$, $s = s(0) + \int_0^b v(t) \, dt = 15 - 4 + 5 = 16$.
At $t = c$, $s = s(0) + \int_0^c v(t) \, dt = 15 - 4 + 5 - 24 = -8$.

15. At $t = a$, where $\frac{dv}{dt}$ is at a maximum (the graph is steepest upward).

16. At $t = c$, where $\frac{dv}{dt}$ is at a maximum (the graph is steepest upward).

17. Distance = Area under curve $= 4\left(\frac{1}{2} \cdot 1 \cdot 2\right) = 4$

(a) Final position = Initial position + Distance $= 2 + 4 = 6$; ends at $x = 6$.

(b) 4 meters

18. **(a)** Positive and negative velocities cancel: the sum of signed areas is zero. Starts and ends at $x = 2$.

(b) Distance = Sum of positive areas $= 4(1 \cdot 1) = 4$ meters

19. **(a)** Final position $= 2 + \int_0^7 v(t)\,dt$

$$= 2 - \frac{1}{2}(1)(2) + \frac{1}{2}(1)(2) + 1(2) + \frac{1}{2}(2)(2) - \frac{1}{2}(2)(1) = 5;$$

ends at $x = 5$.

(b) $\int_0^7 |v(t)|\,dt = \frac{1}{2}(1)(2) + \frac{1}{2}(1)(2) + 1(2) + \frac{1}{2}(2)(2) + \frac{1}{2}(2)(1) = 7$ meters

20. **(a)** Final position $= 2 + \int_0^{10} v(t)\,dt$

$$= 2 + \frac{1}{2}(2)(3) - \frac{1}{2}(1)(3) - (3)(3) - \frac{1}{2}(1)(3) + \frac{1}{2}(3)(3) = -2.5;$$

ends at $x = -2.5$.

(b) Distance $= \int_0^{10} |v(t)|\,dt$

$$= \frac{1}{2}(2 \cdot 3) + \frac{1}{2}(1)(3) + 3(3) + \frac{1}{2}(1)(3) + \frac{1}{2}(3)(3) = 19.5 \text{ meters}$$

21. $\int_0^{10} 27.08 \cdot e^{t/25}\,dt = 27.08\left[25e^{t/25}\right]_0^{10} = 27.08[25e^{0.4} - 25]$ ≈ 332.965 billion barrels

22. $\int_0^{24}\left[3.9 - 2.4\sin\left(\frac{\pi t}{12}\right)\right]dt = \left[3.9t + \frac{28.8}{\pi}\cos\left(\frac{\pi t}{12}\right)\right]_0^{24}$ $= \left[\left(93.6 + \frac{28.8}{\pi}\right) - \frac{28.8}{\pi}\right] = 93.6$ kilowatt-hours

23. **(a)** Solve $10{,}000(2 - r) = 0$: $r = 2$ miles.

(b) Width $= \Delta r$, Length $= 2\pi r$: Area $= 2\pi r\Delta r$

(c) Population $=$ Population density $\times$ Area

(d) $\int_0^2 10{,}000(2 - r)(2\pi r)\,dr = 20{,}000\pi\int_0^2 (2r - r^2)\,dr$

$$= 20{,}000\pi\left[r^2 - \frac{1}{3}r^3\right]_0^2 = 20{,}000\pi\left[\left(4 - \frac{8}{3}\right) - 0\right] = \frac{80{,}000}{3}\pi \approx 83{,}776$$

24. **(a)** Width $= \Delta r$, Length $= 2\pi r$: Area $= 2\pi r\Delta r$

(b) Volume per second

$=$ Inches per second $\times$ Cross section area

$8(10 - r^2)\frac{\text{in.}}{\text{sec}} \cdot (2\pi r)\Delta r \text{ in}^2 = \text{flow in } \frac{\text{in}^3}{\text{sec}}$

(c) $\int_0^3 8(10 - r^2)(2\pi r)\,dr = 16\pi\int_0^3 (10r - r^3)\,dr$

$$= 16\pi\left[5r^2 - \frac{1}{4}r^4\right]_0^3 = 16\pi\left[\left(45 - \frac{81}{4}\right) - 0\right] = 396\pi\ \frac{\text{in}^3}{\text{sec}} \approx 1244.07\ \frac{\text{in}^3}{\text{sec}}$$

25. (Answers may vary.)
Plot the speeds vs. time. Connect the points and find the area under the line graph. The definite integral also gives the area under the curve.

26. **(a)** Sum of numbers in Sales column $= 797.5$ thousand

(b) Enter the table in a graphing calculator and use QuadReg: $B(x) = 1.6x^2 + 2.3x + 5.0$.

(c) $\int_0^{11} (1.6x^2 + 2.3x + 5.0)\,dx$

$$= \left[\frac{1.6}{3}x^3 + \frac{2.3}{2}x^2 + 5.0x\right]_0^{11} \approx 904.02 \text{ thousand}$$

(d) The answer in (a) corresponds to the area of left hand rectangles. These rectangles lie under the curve $B(x)$. The answer in (c) corresponds to the area under the curve. This area is greater than the area of rectangles.

27. **(a)** $\int_{-0.5}^{10.5} (1.6x^2 + 2.3x + 5.0)\,dx$

$$= \left[\frac{1.6}{3}x^3 + \frac{2.3}{2}x^2 + 5.0x\right]_{-0.5}^{10.5} \approx 798.97 \text{ thousand}$$

(b) The answer in (a) corresponds to the area of midpoint rectangles. The curve now gives a better approximation since part of each rectangle is above the curve and part is below.

28. Treat 6 P.M. as 18 o'clock:

$$\frac{b - a}{2n}\left[f(x_0) + \sum_{i=1}^{n-1} 2f(x_i) + f(x_n)\right]$$

$$= \frac{18 - 8}{2(10)}[120 + 2(110) + 2(115) + 2(115) + 2(119) + 2(120) + 2(120) + 2(115) + 2(112) + 2(110) + 121]$$

$$= 1156.5$$

29. $F(x) = kx$; $6 = k(3)$, so $k = 2$ and $F(x) = 2x$.

(a) $F(9) = 2(9) = 18$N

(b) $W = \int_0^9 F(x)\,dx = \int_0^9 2x\,dx = \left[x^2\right]_0^9 = 81 \text{ N} \cdot \text{cm}$

30. $F(x) = kx$; $10{,}000 = k(1)$, so $k = 10{,}000$.

(a) $W = \int_0^d kx\,dx = \left[\frac{1}{2}kx^2\right]_0^d = \frac{1}{2}kd^2 = \frac{1}{2}(10{,}000)(0.5)^2$ $= 1250$ inch-pounds

(b) For total distance: $W = \frac{1}{2}(10{,}000)(1)^2 = 5000$

For second half of distance:

$W = 5000 - 1250 = 3750$ inch-pounds

31. $$\frac{(12 - 0)}{2(12)}[0.04 + 2(0.04) + 2(0.05) + 2(0.06) + 2(0.05) + 2(0.04) + 2(0.04) + 2(0.05) + 2(0.04) + 2(0.06) + 2(0.06) + 2(0.05) + 0.05] = 0.585$$

The overall rate, then, is $\frac{0.585}{12} = 0.04875$.

32. $\frac{(12-0)}{2(12)}[3.6 + 2(4.0) + 2(3.1) + 2(2.8) + 2(2.8) + 2(3.2)$
$+ 2(3.3) + 2(3.1) + 2(3.2) + 2(3.4) + 2(3.4)$
$+ 2(3.9) + 4.0] = 40$ thousandths or 0.040

33. **(a)** $\bar{x} = \frac{M_y}{M} = \frac{\sum m_k x_k}{\sum m_k}$. Taking $dm = \delta dA$ as m_k and letting $dA \to 0$, $k \to \infty$ yields $\frac{\int x\,dm}{\int dm}$.

(b) $\bar{y} = \frac{M_y}{M} = \frac{\sum m_k y}{\sum m_k}$. Taking $dm = \delta dA$ as m_k and letting $dA \to 0$, $k \to \infty$ yields $\frac{\int y\,dm}{\int dm}$.

34. By symmetry, $\bar{x} = 0$. For $\bar{y}$, use horizontal strips:

$$\bar{y} = \frac{\int y\,dm}{\int dm} = \frac{\int y\,\delta\,dA}{\int \delta\,dA} = \frac{\int y\,dA}{\int dA}$$
$$= \frac{\int_0^4 y(2\sqrt{y})\,dy}{\int_0^4 2\sqrt{y}\,dy}$$
$$= \frac{2\left[\frac{2}{5}y^{5/2}\right]_0^4}{2\left[\frac{2}{3}y^{3/2}\right]_0^4}$$
$$= \frac{12}{5}$$

35. By symmetry, $y = 0$. For x, use vertical strips:

$$x = \frac{\int x\,dm}{\int dm} = \frac{\int x\delta\,dA}{\int \delta\,dA} = \frac{\int x\,dA}{\int dA}$$
$$= \frac{\int_0^2 x(2x)\,dx}{\int_0^2 2x\,dx}$$
$$= \frac{\left[\frac{2}{3}x^3\right]_0^2}{\left[x^2\right]_0^2}$$
$$= \frac{4}{3}$$

■ Section 7.2 Areas in the Plane (pp. 374–382)

Exploration 1 A Family of Butterflies

1. For $k = 1$:
$$\int_0^{\pi}[(2 - \sin x) - \sin x]\,dx = \int_0^{\pi}(2 - 2\sin x)\,dx$$
$$= 2x + 2\cos x\Big]_0^{\pi}$$
$$= 2\pi - 4$$

For $k = 2$:
$$\int_0^{\pi/2}[(4 - 2\sin 2x) - (2\sin 2x)]\,dx$$
$$= \int_0^{\pi/2}(4 - 4\sin 2x)\,dx$$
$$= 4x + 2\cos 2x\Big]_0^{\pi/2} = 2\pi - 4$$

2. It appears that the areas for $k \geq 3$ will continue to be $2\pi - 4$.

3. $$A_k = \int_0^{\pi/k}[(2k - k\sin kx) - k\sin kx]\,dx$$
$$= \int_0^{\pi/k}(2k - 2k\sin kx)\,dx$$

If we make the substitution $u = kx$, then $du = k\,dx$ and the u-limits become 0 to π. Thus,

$$A_k = \int_0^{\pi/k}(2k - 2k\sin kx)\,dx$$
$$= \int_0^{\pi/k}(2 - 2\sin kx)k\,dx$$
$$= \int_0^{\pi}(2 - 2\sin u)\,du.$$

4. $2\pi - 4$

5. Because the amplitudes of the sine curves are k, the kth butterfly stands $2k$ units tall. The vertical edges alone have lengths $(2k)$ that increase without bound, so the perimeters are tending to infinity.

Quick Review 7.2

1. $\int_0^{\pi}\sin x\,dx = \Big[-\cos x\Big]_0^{\pi} = -[-1 - 1] = 2$

2. $\int_0^1 e^{2x}\,dx = \left[\frac{1}{2}e^{2x}\right]_0^1 = \frac{1}{2}(e^2 - 1) \approx 3.195$

3. $\int_{-\pi/4}^{\pi/4}\sec^2 x\,dx = \Big[\tan x\Big]_{-\pi/4}^{\pi/4} = 1 - (-1) = 2$

4. $\int_0^2(4x - x^3)\,dx = \left[2x^2 - \frac{1}{4}x^4\right]_0^2 = (8 - 4) - 0 = 4$

5. $\int_{-3}^{3}\sqrt{9 - x^2}\,dx = \frac{9\pi}{2}$ (This is half the area of a circle of radius 3.)

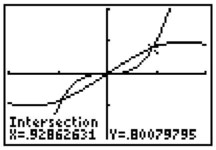
6. Solve $x^2 - 4x = x + 6$.
$x^2 - 5x - 6 = 0$
$(x - 6)(x + 1) = 0$
$x = 6$ or $x = -1$
$y = 6 + 6 = 12$ or $y = -1 + 6 = 5$
$(6, 12)$ and $(-1, 5)$

7. Solve $e^x = x + 1$. From the graphs, it appears that e^x is always greater than or equal to $x + 1$, so that if they are ever equal, this is when $e^x - (x + 1)$ is at a minimum.
$\frac{d}{dx}[e^x - (x + 1)] = e^x - 1$ is zero when $e^x = 1$, i.e., when $x = 0$. Test: $e^0 = 0 + 1 = 1$. So the solution is $(0, 1)$.

8. Inspection of the graphs shows two intersection points: $(0, 0)$, and $(\pi, 0)$. Check: $0^2 - \pi \cdot 0 = \sin 0 = 0$ and $\pi^2 - \pi^2 = \sin \pi = 0$.

9. Solve $\frac{2x}{x^2 + 1} = x^3$.
$(0, 0)$ is a solution. Now divide by x.
$\frac{2}{x^2 + 1} = x^2$
$2 = x^4 + x^2$
$x^4 + x^2 - 2 = 0$
$x^2 = \frac{-1 \pm \sqrt{1 + 8}}{2} = -2$ or 1
Throw out the negative solution.
$x = \pm 1$
$y = x^3 = \pm 1$
$(0, 0)$, $(-1, -1)$ and $(1, 1)$

10. Use the intersect function on a graphing calculator:



$[-2, 2]$ by $[-2, 2]$

$(-0.9286, -0.8008)$, $(0, 0)$, and $(0.9286, 0.8008)$

Section 7.2 Exercises

1. $\int_0^{\pi} (1 - \cos^2 x)\, dx = \left[\frac{1}{2}x - \frac{1}{4}\sin 2x\right]_0^{\pi} = \frac{\pi}{2}$

2. Use symmetry:

$$2\int_0^{\pi/3}\left(\frac{1}{2}\sec^2 t + 4\sin^2 t\right) dt = \int_0^{\pi/3}(\sec^2 t + 8\sin^2 t)\, dt$$
$$= \left[\tan t + 4t - 2\sin 2t\right]_0^{\pi/3}$$
$$= \left(\sqrt{3} + \frac{4\pi}{3} - \sqrt{3}\right) - 0$$
$$= \frac{4\pi}{3}$$

3. $\int_0^1 (y^2 - y^3)\, dy = \left[\frac{1}{3}y^3 - \frac{1}{4}y^4\right]_0^1 = \frac{1}{12}$

4. $\int_0^1 [(12y^2 - 12y^3) - (2y^2 - 2y)]\, dy$
$$= \int_0^1 (-12y^3 + 10y^2 + 2y)\, dy$$
$$= \left[-3y^4 + \frac{10}{3}y^3 + y^2\right]_0^1$$
$$= -3 + \frac{10}{3} + 1 = \frac{4}{3}$$

5. Use the region's symmetry:

$$2\int_0^2 [2x^2 - (x^4 - 2x^2)]\, dx = 2\int_0^2 (-x^4 + 4x^2)\, dx$$
$$= 2\left[-\frac{1}{5}x^5 + \frac{4}{3}x^3\right]_0^2$$
$$= 2\left[\left(-\frac{32}{5} + \frac{32}{3}\right) - 0\right] = \frac{128}{15}$$

6. Use the region's symmetry:

$$2\int_0^1 (x^2 + 2x^4)\, dx = 2\left[\frac{1}{3}x^3 + \frac{2}{5}x^5\right]_0^1 = 2\left(\frac{1}{3} + \frac{2}{5}\right) = \frac{22}{15}$$

7. Integrate with respect to y:

$$\int_0^1 (2\sqrt{y} - y)\, dy = \left[\frac{4}{3}y^{3/2} - \frac{1}{2}y^2\right]_0^1$$
$$= \left(\frac{4}{3} - \frac{1}{2}\right) - 0 = \frac{5}{6}$$

8. Integrate with respect to y:

$$\int_0^1 [(2 - y) - \sqrt{y}]\, dy$$
$$= \left[2y - \frac{1}{2}y^2 - \frac{2}{3}y^{3/2}\right]_0^1 = \left(2 - \frac{1}{2} - \frac{2}{3}\right) - 0 = \frac{5}{6}$$

9. Integrate in two parts:

$$\int_{-2}^0 [(2x^3 - x^2 - 5x) - (-x^2 + 3x)]\, dx +$$
$$\int_0^2 [(-x^2 + 3x) - (2x^3 - x^2 - 5x)]\, dx$$
$$= \int_{-2}^0 (2x^3 - 8x)\, dx + \int_0^2 (-2x^3 + 8x)\, dx$$
$$= \left[\frac{1}{2}x^4 - 4x^2\right]_{-2}^0 + \left[-\frac{1}{2}x^4 + 4x^2\right]_0^2$$
$$= [0 - (8 - 16)] + [(-8 + 16) - 0] = 16$$

10. Integrate in three parts:

$$\int_{-2}^{-1}[(-x+2)-(4-x^2)]\,dx\;+$$

$$\int_{-1}^{2}[(4-x^2)-(-x+2)]\,dx\;+$$

$$\int_{2}^{3}[(-x+2)-(4-x^2)]\,dx$$

$$=\int_{-2}^{-1}(x^2-x-2)\,dx+\int_{-1}^{2}(-x^2+x+2)\,dx\;+$$

$$\int_{2}^{3}(x^2-x-2)\,dx$$

$$=\left[\frac{1}{3}x^3-\frac{1}{2}x^2-2x\right]_{-2}^{-1}+\left[-\frac{1}{3}x^3+\frac{1}{2}x^2+2x\right]_{-1}^{2}+$$

$$\left[\frac{1}{3}x^3-\frac{1}{2}x^2-2x\right]_{2}^{3}$$

$$=\left[\left(-\frac{1}{3}-\frac{1}{2}+2\right)-\left(-\frac{8}{3}-2+4\right)\right]+$$

$$\left[\left(-\frac{8}{3}+2+4\right)-\left(\frac{1}{3}+\frac{1}{2}-2\right)\right]+$$

$$\left[\left(9-\frac{9}{2}-6\right)-\left(\frac{8}{3}-2-4\right)\right]$$

$$=\frac{49}{6}=8\frac{1}{6}$$

11. Solve $x^2-2=2$: $x^2=4$, so the curves intersect at $x=\pm 2$.

$$\int_{-2}^{2}[2-(x^2-2)]\,dx=\int_{-2}^{2}(4-x^2)\,dx$$

$$=\left[4x-\frac{1}{3}x^3\right]_{-2}^{2}=\left(8-\frac{8}{3}\right)-\left(-8+\frac{8}{3}\right)=\frac{32}{3}=10\frac{2}{3}$$

12. Solve $2x-x^2=-3$: $x^2-2x-3=(x-3)(x+1)=0$, so the curves intersect at $x=-1$ and $x=3$.

$$\int_{-1}^{3}(2x-x^2+3)\,dx=\left[x^2-\frac{1}{3}x^3+3x\right]_{-1}^{3}$$

$$=(9-9+9)-\left(1+\frac{1}{3}-3\right)$$

$$=\frac{32}{3}=10\frac{2}{3}$$

13. Solve $7-2x^2=x^2+4$: $x^2=1$, so the curves intersect at $x=\pm 1$.

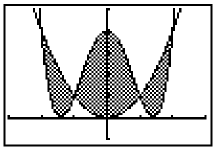
$$\int_{-1}^{1}[(7-2x^2)-(x^2+4)]\,dx=\int_{-1}^{1}(-3x^2+3)\,dx$$

$$=3\int_{-1}^{1}(1-x^2)\,dx$$

$$=3\left[x-\frac{1}{3}x^3\right]_{-1}^{1}$$

$$=3\left[\frac{2}{3}-\left(-\frac{2}{3}\right)\right]=4$$

14.



$[-3, 3]$ by $[-1, 5]$

The curves intersect at $x=\pm 1$ and $x=\pm 2$. Use the region's symmetry:

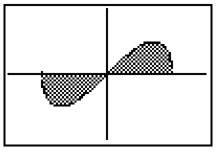
$$2\int_{0}^{1}[(x^4-4x^2+4)-x^2]\,dx+2\int_{1}^{2}[x^2-(x^4-4x^2+4)]\,dx$$

$$=2\int_{0}^{1}(x^4-5x^2+4)\,dx+2\int_{1}^{2}(-x^4+5x^2-4)\,dx$$

$$=2\left[\frac{1}{5}x^5-\frac{5}{3}x^3+4x\right]_{0}^{1}+2\left[-\frac{1}{5}x^5+\frac{5}{3}x^3-4x\right]_{1}^{2}$$

$$=2\left[\frac{1}{5}-\frac{5}{3}+4\right]+2\left[\left(-\frac{32}{5}+\frac{40}{3}-8\right)-\left(-\frac{1}{5}+\frac{5}{3}-4\right)\right]=8$$

15.



$\left[-\frac{3}{2}a, \frac{3}{2}a\right]$ by $[-a^2, a^2]$

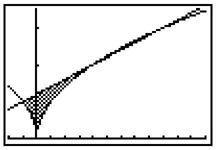
The curves intersect at $x=0$ and $x=\pm a$. Use the region's symmetry:

$$2\int_{0}^{a}x\sqrt{a^2-x^2}\,dx=2\left[-\frac{1}{3}(a^2-x^2)^{3/2}\right]_{0}^{a}$$

$$=2\left[0-\left(-\frac{1}{3}a^3\right)\right]$$

$$=\frac{2}{3}a^3$$

16.



[−2, 12] by [0, 3.5]

The curves intersect at three points:
$x = -1$, $x = 4$ and $x = 9$.
Because of the absolute value sign, break the integral up at $x = 0$ also:

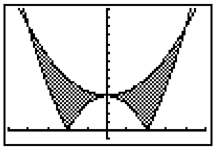
$$\int_{-1}^{0}\left(\frac{x+6}{5} - \sqrt{-x}\right)dx + \int_{0}^{4}\left(\frac{x+6}{5} - \sqrt{x}\right)dx + \int_{4}^{9}\left(\sqrt{x} - \frac{x+6}{5}\right)dx$$

$$= \left[\frac{\frac{1}{2}x^2 + 6x}{5} + \frac{2}{3}(-x)^{3/2}\right]_{-1}^{0} + \left[\frac{\frac{1}{2}x^2 + 6x}{5} - \frac{2}{3}x^{3/2}\right]_{0}^{4} + \left[\frac{2}{3}x^{3/2} - \frac{\frac{1}{2}x^2 + 6x}{5}\right]_{4}^{9}$$

$$= \left[0 - \left(-\frac{11}{10} + \frac{2}{3}\right)\right] + \left[\left(\frac{32}{5} - \frac{16}{3}\right) - 0\right] + \left[\left(18 - \frac{189}{10}\right) - \left(\frac{16}{3} - \frac{32}{5}\right)\right]$$

$$= \frac{13}{30} + \frac{16}{15} + \frac{1}{6} = \frac{5}{3} = 1\frac{2}{3}$$

17.



[−5, 5] by [−1, 14]

The curves intersect at $x = 0$ and $x = \pm 4$. Because of the absolute value sign, break the integral up at $x = \pm 2$ also (where $|x^2 - 4|$ turns the corner). Use the graph's symmetry:

$$2\int_{0}^{2}\left[\left(\frac{x^2}{2} + 4\right) - (4 - x^2)\right]dx + 2\int_{2}^{4}\left[\left(\frac{x^2}{2} + 4\right) - (x^2 - 4)\right]dx$$

$$= 2\int_{0}^{2}\frac{3x^2}{2}\,dx + 2\int_{2}^{4}\left(-\frac{x^2}{2} + 8\right)dx$$

$$= 2\left[\frac{x^3}{2}\right]_{0}^{2} + 2\left[-\frac{x^3}{6} + 8x\right]_{2}^{4}$$

$$= 2[4] + 2\left[\left(-\frac{32}{3} + 32\right) - \left(-\frac{4}{3} + 16\right)\right] = \frac{64}{3} = 21\frac{1}{3}$$

18. Solve $y^2 = y + 2$: $y^2 - y - 2 = (y - 2)(y + 1) = 0$, so the curves intersect at $y = -1$ and $y = 2$.

$$\int_{-1}^{2}(y + 2 - y^2)\,dy = \left[\frac{1}{2}y^2 + 2y - \frac{1}{3}y^3\right]_{-1}^{2}$$

$$= \left(2 + 4 - \frac{8}{3}\right) - \left(\frac{1}{2} - 2 + \frac{1}{3}\right)$$

$$= \frac{9}{2} = 4\frac{1}{2}$$

19. Solve for x: $x = \frac{y^2}{4} - 1$ and $x = \frac{y}{4} + 4$.

Now solve $\frac{y^2}{4} - 1 = \frac{y}{4} + 4$: $\frac{y^2}{4} - \frac{y}{4} - 5 = 0$,

$y^2 - y - 20 = (y - 5)(y + 4) = 0$.

The curves intersect at $y = -4$ and $y = 5$.

$$\int_{-4}^{5}\left[\left(\frac{y}{4} + 4\right) - \left(\frac{y^2}{4} - 1\right)\right]dy$$

$$= \int_{-4}^{5}\left(-\frac{y^2}{4} + \frac{y}{4} + 5\right)dy$$

$$= \left[-\frac{y^3}{12} + \frac{y^2}{8} + 5y\right]_{-4}^{5}$$

$$= \left(-\frac{125}{12} + \frac{25}{8} + 25\right) - \left(\frac{16}{3} + 2 - 20\right) = \frac{243}{8} = 30\frac{3}{8}$$

20. Solve for x: $x = y^2$ and $x = 3 - 2y^2$. Now solve

$y^2 = 3 - 2y^2$: $y^2 = 1$, so the curves intersect at $y = \pm 1$.

Use the region's symmetry:

$$2\int_{0}^{1}(3 - 2y^2 - y^2)\,dy = 2\int_{0}^{1}(3 - 3y^2)\,dy$$

$$= 6\int_{0}^{1}(1 - y^2)\,dy$$

$$= 6\left[y - \frac{1}{3}y^3\right]_{0}^{1}$$

$$= 6\left[\left(1 - \frac{1}{3}\right) - 0\right] = 4$$

21. Solve for x: $x = -y^2$ and $x = 2 - 3y^2$.

Now solve $-y^2 = 2 - 3y^2$: $y^2 = 1$, so the curves intersect at $y = \pm 1$. Use the region's symmetry:

$$2\int_{0}^{1}(2 - 3y^2 + y^2)\,dy = 2\int_{0}^{1}(2 - 2y^2)\,dy$$

$$= 4\int_{0}^{1}(1 - y^2)\,dy = 4\left[y - \frac{1}{3}y^3\right]_{0}^{1} = 4\left[\left(1 - \frac{1}{3}\right) - 0\right] = \frac{8}{3}$$

22. Solve for y: $y = 4 - 4x^2$ and $y = x^4 - 1$.

Now solve $4 - 4x^2 = x^4 - 1$:

$x^4 + 4x^2 - 5 = (x^2 - 1)(x^2 + 5) = 0$.

The curves intersect at $x = \pm 1$.

Use the region's symmetry:

$$2\int_0^1 [(4 - 4x^2) - (x^4 - 1)]\,dx$$
$$= 2\int_0^1 (-x^4 - 4x^2 + 5)\,dx$$
$$= 2\left[-\frac{1}{5}x^5 - \frac{4}{3}x^3 + 5x\right]_0^1$$
$$= 2\left[\left(-\frac{1}{5} - \frac{4}{3} + 5\right) - 0\right]$$
$$= \frac{104}{15} = 6\frac{14}{15}$$

23. Solve for x: $x = 3 - y^2$ and $x = -\frac{y^2}{4}$.

Now solve $3 - y^2 = -\frac{y^2}{4}$: $y^2 = 4$,

so the curves intersect at $y = \pm 2$.

Use the region's symmetry:

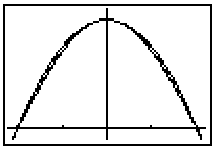
$$2\int_0^2 \left(3 - y^2 + \frac{y^2}{4}\right) dy = 2\int_0^2 \left(3 - \frac{3y^2}{4}\right) dy$$
$$= 2\left[3y - \frac{y^3}{4}\right]_0^2$$
$$= 2(6 - 2) - 0 = 8$$

24. $$\int_0^\pi (2\sin x - \sin 2x)\,dx = \left[-2\cos x + \frac{1}{2}\cos 2x\right]_0^\pi$$
$$= \left[\left(2 + \frac{1}{2}\right) - \left(-2 + \frac{1}{2}\right)\right] = 4$$

25. Use the region's symmetry:

$$2\int_0^{\pi/3} (8\cos x - \sec^2 x)\,dx = 2\Big[8\sin x - \tan x\Big]_0^{\pi/3}$$
$$= 2[(4\sqrt{3} - \sqrt{3}) - 0] = 6\sqrt{3}$$

26.

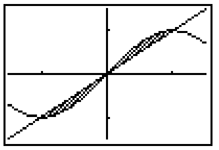


[−1.1, 1.1] by [−0.1, 1.1]

The curves intersect at $x = 0$ and $x = \pm 1$, but they do not cross at $x = 0$.

$$2\int_0^1 \left[1 - x^2 - \cos\left(\frac{\pi x}{2}\right)\right] dx$$
$$= 2\left[x - \frac{1}{3}x^3 - \frac{2}{\pi}\sin\left(\frac{\pi x}{2}\right)\right]_0^1$$
$$= 2\left[\left(1 - \frac{1}{3} - \frac{2}{\pi}\right) - 0\right] = \frac{4}{3} - \frac{4}{\pi} \approx 0.0601$$

27.



[−1.5, 1.5] by [−1.5, 1.5]

The curves intersect at $x = 0$ and $x = \pm 1$. Use the area's symmetry:

$$2\int_0^1 \left[\sin\left(\frac{\pi x}{2}\right) - x\right] dx = 2\left[-\frac{2}{\pi}\cos\left(\frac{\pi x}{2}\right) - \frac{1}{2}x^2\right]_0^1$$
$$= 2\left[-\frac{1}{2} - \left(-\frac{2}{\pi}\right)\right]$$
$$= \frac{4 - \pi}{\pi} \approx 0.273$$

28. Use the region's symmetry, and simplify before integrating:

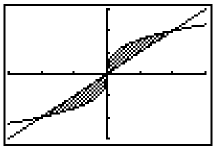
$$2\int_0^{\pi/4} (\sec^2 x - \tan^2 x)\,dx$$
$$= 2\int_0^{\pi/4} [\sec^2 x - (\sec^2 x - 1)]\,dx$$
$$= 2\int_0^{\pi/4} dx = 2\Big[x\Big]_0^{\pi/4} = \frac{\pi}{2}$$

29. Use the region's symmetry:

$$2\int_0^{\pi/4} (\tan^2 y + \tan^2 y)\,dy = 4\int_0^{\pi/4} \tan^2 y\,dy$$
$$= 4\Big[\tan y - y\Big]_0^{\pi/4}$$
$$= 4\left[\left(1 - \frac{\pi}{4}\right) - 0\right]$$
$$= 4 - \pi \approx 0.858$$

30. $$\int_0^{\pi/2} 3\sin y\sqrt{\cos y}\,dy = 3\left[-\frac{2}{3}(\cos y)^{3/2}\right]_0^{\pi/2}$$
$$= 3\left[0 - \left(-\frac{2}{3}\right)\right] = 2$$

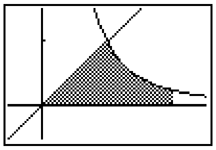
31. Solve for x: $x = y^3$ and $x = y$.



[−1.5, 1.5] by [−1.5, 1.5]

The curves intersect at $x = 0$ and $x = \pm 1$. Use the area's symmetry: $2\int_0^1 (y - y^3)\,dy = 2\left[\frac{1}{2}y^2 - \frac{1}{4}y^4\right]_0^1 = \frac{1}{2}$

32.



$[-0.5, 2.5]$ by $[-0.5, 1.5]$

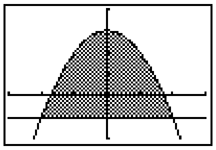
$y = x$ and $y = \frac{1}{x^2}$ intersect at $x = 1$. Integrate in two parts:

$$\int_0^1 x\,dx + \int_1^2 \frac{1}{x^2}\,dx = \left[\frac{1}{2}x^2\right]_0^1 + \left[-\frac{1}{x}\right]_1^2$$
$$= \frac{1}{2} + \left[-\frac{1}{2} - (-1)\right] = 1$$

33. The curves intersect when $\sin x = \cos x$, i.e., at $x = \frac{\pi}{4}$.

$$\int_0^{\pi/4} (\cos x - \sin x)\,dx = \left[\sin x + \cos x\right]_0^{\pi/4}$$
$$= \sqrt{2} - 1 \approx 0.414$$

34.



$[-3, 3]$ by $[-2, 4]$

(a) The curves intersect at $x = \pm 2$.

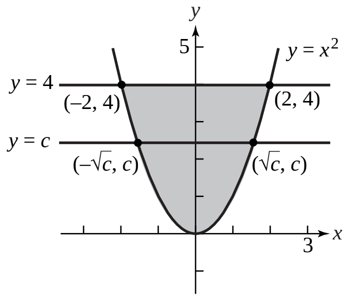
Use the region's symmetry:

$$2\int_0^2 (3 - x^2 + 1)\,dx = 2\int_0^2 (4 - x^2)\,dx$$
$$= 2\left[4x - \frac{1}{3}x^3\right]_0^2$$
$$= 2\left[\left(8 - \frac{8}{3}\right) - 0\right] = \frac{32}{3}$$

(b) Solve $y = 3 - x^2$ for x: $x = \pm\sqrt{3 - y}$. The y-intercepts are -1 and 3.

$$\int_{-1}^3 2\sqrt{3 - y}\,dy \qquad = 2\left[-\frac{2}{3}(3 - y)^{3/2}\right]_{-1}^3$$
$$= 2\left[0 - \left(-\frac{16}{3}\right)\right] = \frac{32}{3}$$

35. (a)



If $y = x^2 = c$, then $x = \pm\sqrt{c}$. So the points are $(-\sqrt{c}, c)$ and $(\sqrt{c}, c)$.

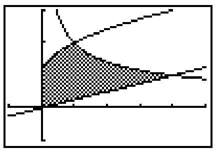
(b) The two areas in Quadrant I, where $x = \sqrt{y}$, are equal:

$$\int_0^c \sqrt{y}\,dy = \int_c^4 \sqrt{y}\,dy$$
$$\left[\frac{2}{3}y^{3/2}\right]_0^c = \left[\frac{2}{3}y^{3/2}\right]_c^4$$
$$\frac{2}{3}c^{3/2} = \frac{2}{3}4^{3/2} - \frac{2}{3}c^{3/2}$$
$$2c^{3/2} = 8$$
$$c^{3/2} = 4$$
$$c = 4^{2/3} = 2^{4/3}$$

(c) Divide the upper right section into a $(4 - c)$-by-$\sqrt{c}$ rectangle and a leftover portion:

$$\int_0^{\sqrt{c}} (c - x^2)\,dx = (4 - c)\sqrt{c} + \int_{\sqrt{c}}^2 (4 - x^2)\,dx$$
$$\left[cx - \frac{1}{3}x^3\right]_0^{\sqrt{c}} = 4\sqrt{c} - c^{3/2} + \left[4x - \frac{1}{3}x^3\right]_{\sqrt{c}}^2$$
$$c^{3/2} - \frac{1}{3}c^{3/2} = 4\sqrt{c} - c^{3/2}$$
$$+ \left[\left(8 - \frac{8}{3}\right) - \left(4\sqrt{c} - \frac{1}{3}c^{3/2}\right)\right]$$
$$\frac{2}{3}c^{3/2} = 4\sqrt{c} - c^{3/2} + \frac{16}{3} - 4\sqrt{c} + \frac{1}{3}c^{3/2}$$
$$\frac{4}{3}c^{3/2} = \frac{16}{3}$$
$$c^{3/2} = 4$$
$$c = 4^{2/3} = 2^{4/3}$$

36.



$[-1, 5]$ by $[-1, 3]$

The key intersection points are at $x = 0$, $x = 1$ and $x = 4$.

Integrate in two parts:

$$\int_0^1 \left(1 + \sqrt{x} - \frac{x}{4}\right)dx + \int_1^4 \left(\frac{2}{\sqrt{x}} - \frac{x}{4}\right)dx$$
$$= \left[x + \frac{2}{3}x^{3/2} - \frac{x^2}{8}\right]_0^1 + \left[4\sqrt{x} - \frac{x^2}{8}\right]_1^4$$
$$= \left(1 + \frac{2}{3} - \frac{1}{8}\right) + \left[(8 - 2) - \left(4 - \frac{1}{8}\right)\right] = \frac{11}{3}$$

37. First find the two areas.

For the triangle, $\frac{1}{2}(2a)(a^2) = a^3$

For the parabola, $2\int_0^a (a^2 - x^2)\,dx = 2\left[a^2x - \frac{1}{3}x^3\right]_0^a = \frac{4}{3}a^3$

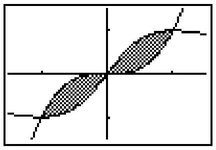
The ratio, then, is $\dfrac{a^3}{\frac{4}{3}a^3} = \dfrac{3}{4}$, which remains constant as a approaches zero.

38. $\int_a^b [2f(x) - f(x)]\,dx = \int_a^b f(x)\,dx$, which we already know equals 4.

39. Neither; both integrals come out as zero because the -1-to-0 and 0-to-1 portions of the integrals cancel each other.

40. Sometimes true, namely when $dA = [f(x) - g(x)]\,dx$ is always nonnegative. This happens when $f(x) \geq g(x)$ over the entire interval.

41.

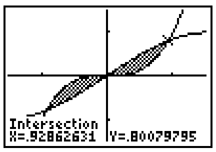


$[-1.5, 1.5]$ by $[-1.5, 1.5]$

The curves intersect at $x = 0$ and $x = \pm 1$. Use the area's symmetry:

$$2\int_0^1 \left(\frac{2x}{x^2+1} - x^3\right)\,dx = 2\left[\ln\left(x^2+1\right) - \frac{1}{4}x^4\right]_0^1$$
$$= 2\ln 2 - \frac{1}{2}$$
$$= \ln 4 - \frac{1}{2} \approx 0.886$$

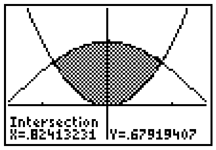
42.



$[-1.5, 1.5]$ by $[-1.5, 1.5]$

The curves intersect at $x = 0$ and $x \approx \pm 0.9286$. Use NINT to find $2\int_0^{0.9286} (\sin x - x^3)\,dx \approx 0.4303$.

43. First graph $y = \cos x$ and $y = x^2$.



$[-1.5, 1.5]$ by $[-0.5, 1.5]$

The curves intersect at $x \approx \pm 0.8241$. Use NINT to find $2\int_0^{0.8241} (\cos x - x^2)\,dx \approx 1.0948$. Multiplying both functions by k will not change the x-value of any intersection point, so the area condition to be met is

$$2 = 2\int_0^{0.8241} (k\cos x - kx^2)\,dx$$
$$\Rightarrow 2 = k \cdot 2\int_0^{0.8241} (\cos x - x^2)\,dx$$
$$\Rightarrow 2 \approx k(1.0948)$$
$$\Rightarrow k \approx 1.8269.$$

44. (a) Solve for y:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$
$$y^2 = b^2\left(1 - \frac{x^2}{a^2}\right)$$
$$y = \pm b\sqrt{1 - \frac{x^2}{a^2}}$$

(b) $\int_{-a}^{a}\left[b\sqrt{1-\frac{x^2}{a^2}} - \left(-b\sqrt{1-\frac{x^2}{a^2}}\right)\right]dx$ or

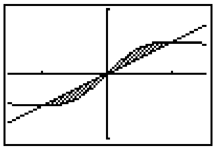
$2\int_{-a}^{a} b\sqrt{1-\frac{x^2}{a^2}}\,dx$ or $4\int_0^a b\sqrt{1-\frac{x^2}{a^2}}\,dx$

(c) Answers may vary.

(d, e) $$2\int_{-a}^{a} b\sqrt{1-\frac{x^2}{a^2}}\,dx = 2b\left[\frac{x}{2}\sqrt{1-\frac{x^2}{a^2}} + \frac{a}{2}\sin^{-1}\frac{x}{a}\right]_{-a}^{a}$$
$$= 2b\left[\frac{a}{2}\sin^{-1}(1) - \frac{a}{2}\sin^{-1}(-1)\right]$$
$$= \pi ab$$

45. By hypothesis, $f(x) - g(x)$ is the same for each region, where $f(x)$ and $g(x)$ represent the upper and lower edges. But then Area $= \int_a^b [f(x) - g(x)]\,dx$ will be the same for each.

46. The curves are shown for $m = \frac{1}{2}$:



$[-1.5, 1.5]$ by $[-1, 1]$

In general, the intersection points are where $\frac{x}{x^2+1} = mx$, which is where $x = 0$ or else $x = \pm\sqrt{\frac{1}{m} - 1}$. Then, because of symmetry, the area is

$$2\int_0^{\sqrt{(1/m)-1}} \left(\frac{x}{x^2+1} - mx\right) dx$$
$$= 2\left[\frac{1}{2}\ln(x^2+1) - \frac{1}{2}mx^2\right]_0^{\sqrt{(1/m)-1}}$$
$$= \ln\left(\frac{1}{m} - 1 + 1\right) -$$
$$m\left(\frac{1}{m} - 1\right) = m - \ln(m) - 1.$$

■ Section 7.3 Volumes
(pp. 383–394)

Exploration 1 Volume by Cylindrical Shells

1. Its height is $f(x_k) = 3x_k - x_k^2$.
2. Unrolling the cylinder, the circumference becomes one dimension of a rectangle, and the height becomes the other. The thickness Δx is the third dimension of a slab with dimensions $2\pi(x_k + 1)$ by $3x_k - x_k^2$ by Δx. The volume is obtained by multiplying the dimensions together.
3. The limit is the definite integral $\int_0^3 2\pi(x+1)(3x - x^2)\,dx$.
4. $\frac{45\pi}{2}$

Exploration 2 Surface Area

1. $\int_a^b 2\pi y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$

   The limit will exist if f and f' are continuous on the interval $[a, b]$.

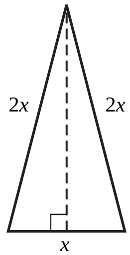
2. $y = \sin x$, so $\frac{dy}{dx} = \cos x$ and

   $$\int_a^b 2\pi y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$$
   $$= \int_0^{\pi} 2\pi \sin x\sqrt{1 + \cos^2 x}\,dx \approx 14.424.$$

3. $y = \sqrt{x}$, so $\frac{dy}{dx} = \frac{1}{2\sqrt{x}}$ and

   $$\int_0^4 2\pi\sqrt{x}\sqrt{1 + \left(\frac{1}{2\sqrt{x}}\right)^2}\,dx \approx 36.177.$$

Quick Review 7.3

1. x^2
2. $s = \frac{x}{\sqrt{2}}$, so Area $= s^2 = \frac{x^2}{2}$.
3. $\frac{1}{2}\pi r^2$ or $\frac{\pi x^2}{2}$
4. $\frac{1}{2}\pi\left(\frac{d}{2}\right)^2$ or $\frac{\pi x^2}{8}$
5. $b = x$ and $h = \frac{\sqrt{3}}{2}x$, so Area $= \frac{1}{2}bh = \frac{\sqrt{3}}{4}x^2$.
6. $b = h = x$, so Area $= \frac{1}{2}bh = \frac{x^2}{2}$.
7. $b = h = \frac{x}{\sqrt{2}}$, so Area $= \frac{1}{2}bh = \frac{x^2}{4}$.
8. 

   $b = x$ and $h = \sqrt{(2x)^2 - \left(\frac{1}{2}x\right)^2} = \frac{\sqrt{15}}{2}x$, so

   Area $= \frac{1}{2}bh = \frac{\sqrt{15}}{4}x^2$.

9. This is a 3-4-5 right triangle. $b = 4x$, $h = 3x$, and Area $= \frac{1}{2}bh = 6x^2$.
10. The hexagon contains six equilateral triangles with sides of length x, so from Exercise 5, Area $= 6\left(\frac{\sqrt{3}}{4}x^2\right) = \frac{3\sqrt{3}}{2}x^2$.

Section 7.3 Exercises

1. In each case, the width of the cross section is $w = 2\sqrt{1 - x^2}$.

   (a) $A = \pi r^2$, where $r = \frac{w}{2}$, so $A(x) = \pi\left(\frac{w}{2}\right)^2 = \pi(1 - x^2)$.

   (b) $A = s^2$, where $s = w$, so $A(x) = w^2 = 4(1 - x^2)$.

   (c) $A = s^2$, where $s = \frac{w}{\sqrt{2}}$, so $A(x) = \left(\frac{w}{\sqrt{2}}\right)^2 = 2(1 - x^2)$.

   (d) $A = \frac{\sqrt{3}}{4}w^2$ (see Quick Review Exercise 5), so $A(x) = \frac{\sqrt{3}}{4}(2\sqrt{1 - x^2})^2 = \sqrt{3}(1 - x^2)$.

2. In each case, the width of the cross section is $w = 2\sqrt{x}$.

(a) $A = \pi r^2$, where $r = \frac{w}{2}$, so $A(x) = \pi\left(\frac{w}{2}\right)^2 = \pi x$.

(b) $A = s^2$, where $s = w$, so $A(x) = w^2 = 4x$.

(c) $A = s^2$, where $s = \frac{w}{\sqrt{2}}$, so $A(x) = \left(\frac{w}{\sqrt{2}}\right)^2 = 2x$.

(d) $A = \frac{\sqrt{3}}{4}w^2$ (see Quick Review Exercise 5), so $A(x) = \frac{\sqrt{3}}{4}(2\sqrt{x})^2 = \sqrt{3}x$.

3. A cross section has width $w = 2\sqrt{x}$ and area $A(x) = s^2 = \left(\frac{w}{\sqrt{2}}\right)^2 = 2x$. The volume is $\int_0^4 2x\,dx = \left[x^2\right]_0^4 = 16$.

4. A cross section has width $w = (2 - x^2) - x^2 = 2 - 2x^2$ and area $A(x) = \pi r^2 = \pi\left(\frac{w}{2}\right)^2 = \pi(1 - x^2)^2$. The volume is

$$\int_{-1}^{1} \pi(1 - x^2)^2\,dx = \pi\int_{-1}^{1}(x^4 - 2x^2 + 1)\,dx = \pi\left[\frac{1}{5}x^5 - \frac{2}{3}x^3 + x\right]_{-1}^{1} = \frac{16}{15}\pi.$$

5. A cross section has width $w = 2\sqrt{1 - x^2}$ and area $A(x) = s^2 = w^2 = 4(1 - x^2)$. The volume is $\int_{-1}^{1} 4(1 - x^2)\,dx = 4\int_{-1}^{1}(1 - x^2)\,dx = 4\left[x - \frac{1}{3}x^3\right]_{-1}^{1} = \frac{16}{3}$.

6. A cross section has width $w = 2\sqrt{1 - x^2}$ and area $A(x) = s^2 = \left(\frac{w}{\sqrt{2}}\right)^2 = 2(1 - x^2)$. The volume is $\int_{-1}^{1} 2(1 - x^2)\,dx = 2\int_{-1}^{1}(1 - x^2)\,dx = 2\left[x - \frac{1}{3}x^3\right]_{-1}^{1} = \frac{8}{3}$.

7. A cross section has width $w = 2\sqrt{\sin x}$.

(a) $A(x) = \frac{\sqrt{3}}{4}w^2 = \sqrt{3}\sin x$, and

$$V = \int_0^{\pi}\sqrt{3}\sin x\,dx = \sqrt{3}\int_0^{\pi}\sin x\,dx = \sqrt{3}\left[-\cos x\right]_0^{\pi} = 2\sqrt{3}.$$

(b) $A(x) = s^2 = w^2 = 4\sin x$, and

$$V = \int_0^{\pi} 4\sin x\,dx = 4\int_0^{\pi}\sin x\,dx = 4\left[-\cos x\right]_0^{\pi} = 8.$$

8. A cross section has width $w = \sec x - \tan x$.

(a) $A(x) = \pi r^2 = \pi\left(\frac{w}{2}\right)^2 = \frac{\pi}{4}(\sec x - \tan x)^2$, and

$$V = \int_{-\pi/3}^{\pi/3}\frac{\pi}{4}(\sec x - \tan x)^2\,dx$$
$$= \frac{\pi}{4}\int_{-\pi/3}^{\pi/3}(\sec^2 x - 2\sec x\tan x + \tan^2 x)\,dx$$
$$= \frac{\pi}{4}\left[\tan x - 2\sec x + \tan x - x\right]_{-\pi/3}^{\pi/3}$$
$$= \frac{\pi}{2}\left[\tan x - \sec x - \frac{1}{2}x\right]_{-\pi/3}^{\pi/3}$$
$$= \frac{\pi}{2}\left[\left(\sqrt{3} - 2 - \frac{\pi}{6}\right) - \left(-\sqrt{3} - 2 + \frac{\pi}{6}\right)\right]$$
$$= \pi\sqrt{3} - \frac{\pi^2}{6}.$$

(b) $A(x) = s^2 = w^2 = (\sec x - \tan x)^2$, and $V = \int_{-\pi/3}^{\pi/3}(\sec x - \tan x)^2\,dx$, which by same method as in part (a) equals $4\sqrt{3} - \frac{2}{3}\pi$.

9. A cross section has width $w = \sqrt{5}y^2$ and area $\pi r^2 = \pi\left(\frac{w}{2}\right)^2 = \frac{5\pi}{4}y^4$. The volume is $\int_0^2 \frac{5\pi}{4}y^4\,dy = \frac{\pi}{4}\left[y^5\right]_0^2 = 8\pi$.

10. A cross section has width $w = 2\sqrt{1 - y^2}$ and area $\frac{1}{2}s^2 = \frac{1}{2}w^2 = 2(1 - y^2)$. The volume is $\int_{-1}^{1} 2(1 - y^2)\,dy = 2\left[y - \frac{1}{3}y^3\right]_{-1}^{1} = \frac{8}{3}$.

11. **(a)** The volume is the same as if the square had moved without twisting: $V = Ah = s^2h$.

(b) Still s^2h: the lateral distribution of the square cross sections doesn't affect the volume. That's Cavalieri's Volume Theorem.

12. Since the diameter of the circular base of the solid extends from $y = \frac{12}{2} = 6$ to $y = 12$, for a diameter of 6 and a radius of 3, the solid has the same cross sections as the right circular cone. The volumes are equal by Cavalieri's Theorem.

13. The solid is a right circular cone of radius 1 and height 2.
$V = \frac{1}{3}Bh = \frac{1}{3}(\pi r^2)h = \frac{1}{3}(\pi 1^2)2 = \frac{2}{3}\pi$

14. The solid is a right circular cone of radius 3 and height 2.
$V = \frac{1}{3}Bh = \frac{1}{3}(\pi r^2)h = \frac{1}{3}(\pi 3^2)2 = 6\pi$

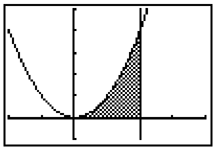
15. A cross section has radius $r = \tan\left(\frac{\pi}{4}y\right)$ and area $A(y) = \pi r^2 = \pi \tan^2\left(\frac{\pi}{4}y\right)$. The volume is

$$\int_0^1 \pi \tan^2\left(\frac{\pi}{4}y\right) dy = \pi\left[\frac{4}{\pi}\tan\left(\frac{\pi}{4}y\right) - y\right]_0^1$$
$$= \pi\left(\frac{4}{\pi} - 1\right)$$
$$= 4 - \pi.$$

16. A cross section has radius $r = \sin x \cos x$ and area $A(x) = \pi r^2 = \pi \sin^2 x \cos^2 x$. The shaded region extends from $x = 0$ to where $\sin x \cos x$ drops back to 0, i.e., where $x = \frac{\pi}{2}$. Now, since $\cos 2x = 2\cos^2 x - 1$, we know $\cos^2 x = \frac{1 + \cos 2x}{2}$ and since $\cos 2x = 1 - 2\sin^2 x$, we know $\sin^2 x = \frac{1 - \cos 2x}{2}$. $\int_0^{\pi/2} \pi \sin^2 x \cos^2 x\, dx$

$$= \pi\int_0^{\pi/2} \frac{1 - \cos 2x}{2} \cdot \frac{1 + \cos 2x}{2}\, dx$$
$$= \frac{\pi}{4}\int_0^{\pi/2} (1 - \cos^2 2x)\, dx = \frac{\pi}{4}\int_0^{\pi/2} \sin^2 2x\, dx$$
$$= \frac{\pi}{4}\int_0^{\pi/2} \frac{1 - \cos 4x}{2}\, dx = \frac{\pi}{8}\int_0^{\pi/2} (1 - \cos 4x)\, dx$$
$$= \frac{\pi}{8}\left[x - \frac{1}{4}\sin 4x\right]_0^{\pi/2} = \frac{\pi}{8}\left[\left(\frac{\pi}{2} - 0\right) - 0\right] = \frac{\pi^2}{16}.$$

17.

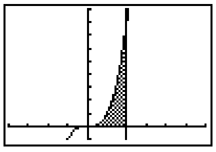


[−2, 4] by [−1, 5]

A cross section has radius $r = x^2$ and area $A(x) = \pi r^2 = \pi x^4$. The volume is

$$\int_0^2 \pi x^4\, dx = \pi\left[\frac{1}{5}x^5\right]_0^2 = \frac{32\pi}{5}.$$

18.

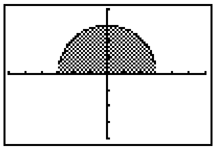


[−4, 6] by [−1, 9]

A cross section has radius $r = x^3$ and area $A(x) = \pi r^2 = \pi x^6$. The volume is

$$\int_0^2 \pi x^6\, dx = \pi\left[\frac{1}{7}x^7\right]_0^2 = \frac{128\pi}{7}.$$

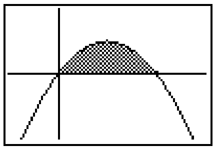
19.



[−6, 6] by [−4, 4]

The solid is a sphere of radius $r = 3$. The volume is $\frac{4}{3}\pi r^3 = 36\pi$.

20.



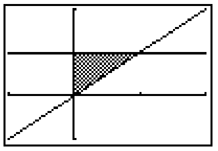
[−0.5, 1.5] by [−0.5, 0.5]

The parabola crosses the line $y = 0$ when $x - x^2 = x(1 - x) = 0$, i.e., when $x = 0$ or $x = 1$. A cross section has radius $r = x - x^2$ and area $A(x) = \pi r^2 = \pi(x - x^2)^2 = \pi(x^2 - 2x^3 + x^4)$.

The volume is

$$\int_0^1 \pi(x^2 - 2x^3 + x^4)\, dx = \pi\left[\frac{1}{3}x^3 - \frac{1}{2}x^4 + \frac{1}{5}x^5\right]_0^1 = \frac{\pi}{30}.$$

21.



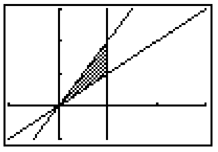
[−1, 2] by [−1, 2]

Use cylindrical shells: A shell has radius y and height y.

The volume is

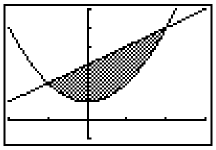
$$\int_0^1 2\pi(y)(y)\, dy = 2\pi\left[\frac{1}{3}y^3\right]_0^1 = \frac{2}{3}\pi.$$

22.



[−1, 3] by [−1, 3]

Use washer cross sections: A washer has inner radius $r = x$, outer radius $R = 2x$, and area $A(x) = \pi(R^2 - r^2) = 3\pi x^2$. The volume is $\int_0^1 3\pi x^2\, dx = 3\pi\left[\frac{1}{3}x^3\right]_0^1 = \pi$.

23. 

$[-2, 3]$ by $[-1, 6]$

The curves intersect when $x^2 + 1 = x + 3$, which is when $x^2 - x - 2 = (x - 2)(x + 1) = 0$, i.e., when $x = -1$ or $x = 2$. Use washer cross sections: a washer has inner radius $r = x^2 + 1$, outer radius $R = x + 3$, and area

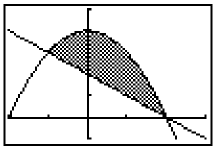
$$\begin{aligned} A(x) &= \pi(R^2 - r^2) \\ &= \pi[(x+3)^2 - (x^2+1)^2] \\ &= \pi(-x^4 - x^2 + 6x + 8). \end{aligned}$$

The volume is

$$\int_{-1}^{2} \pi(-x^4 - x^2 + 6x + 8)\,dx$$

$$= \pi\left[-\frac{1}{5}x^5 - \frac{1}{3}x^3 + 3x^2 + 8x\right]_{-1}^{2}$$

$$= \pi\left[\left(-\frac{32}{5} - \frac{8}{3} + 12 + 16\right) - \left(\frac{1}{5} + \frac{1}{3} + 3 - 8\right)\right] = \frac{117\pi}{5}.$$

24. 

$[-2, 3]$ by $[-1, 5]$

The curves intersect when $4 - x^2 = 2 - x$, which is when $x^2 - x - 2 = (x - 2)(x + 1) = 0$, i.e., when $x = -1$ or $x = 2$. Use washer cross sections: a washer has inner radius $r = 2 - x$, outer radius $R = 4 - x^2$, and area

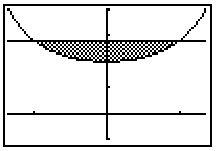
$$\begin{aligned} A(x) &= \pi(R^2 - r^2) \\ &= \pi[(4 - x^2)^2 - (2 - x)^2] \\ &= \pi(12 + 4x - 9x^2 + x^4). \end{aligned}$$

The volume is

$$\int_{-1}^{2} \pi(12 + 4x - 9x^2 + x^4)\,dx$$

$$= \pi\left[12x + 2x^2 - 3x^3 + \frac{1}{5}x^5\right]_{-1}^{2}$$

$$= \pi\left[\left(24 + 8 - 24 + \frac{32}{5}\right) - \left(-12 + 2 + 3 - \frac{1}{5}\right)\right] = \frac{108\pi}{5}.$$

25. 

$\left[-\frac{\pi}{3}, \frac{\pi}{3}\right]$ by $[-0.5, 2]$

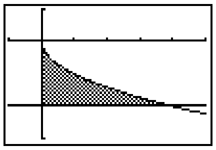
Use washer cross sections: a washer has inner radius $r = \sec x$, outer radius $R = \sqrt{2}$, and area $A(x) = \pi(R^2 - r^2) = \pi(2 - \sec^2 x)$.

The volume is

$$\int_{-\pi/4}^{\pi/4} \pi(2 - \sec^2 x)\,dx = \pi\Big[2x - \tan x\Big]_{-\pi/4}^{\pi/4}$$

$$= \pi\left[\left(\frac{\pi}{2} - 1\right) - \left(-\frac{\pi}{2} + 1\right)\right]$$

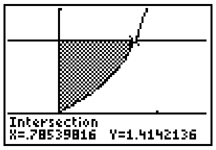
$$= \pi^2 - 2\pi.$$

26. 

$[-1, 5]$ by $[-3, 1]$

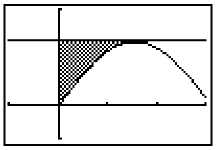
The curves intersect where $-\sqrt{x} = -2$, which is where $x = 4$. Use washer cross sections: a washer has inner radius $r = \sqrt{x}$, outer radius $R = 2$, and area $A(x) = \pi(R^2 - r^2) = \pi(4 - x)$.

The volume is $\displaystyle\int_0^4 \pi(4 - x)\,dx = \pi\left[4x - \frac{1}{2}x^2\right]_0^4 = 8\pi$

27. 

$[-0.5, 1.5]$ by $[-0.5, 2]$

The curves intersect at $x \approx 0.7854$. A cross section has radius $r = \sqrt{2} - \sec x \tan x$ and area $A(x) = \pi r^2 = \pi(\sqrt{2} - \sec x \tan x)^2$. Use NINT to find $\displaystyle\int_0^{0.7854} \pi(\sqrt{2} - \sec x \tan x)^2\,dx \approx 2.301.$

28. 

$[-1, 3]$ by $[-1, 3]$

The curve and horizontal line intersect at $x = \frac{\pi}{2}$. A cross section has radius $2 - 2\sin x$ and area

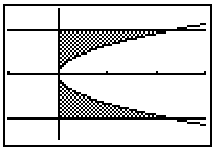
$A(x) = \pi r^2 = 4\pi(1 - \sin x)^2 = 4\pi(1 - 2\sin x + \sin^2 x).$

The volume is

$$\int_0^{\pi/2} 4\pi(1 - 2\sin x + \sin^2 x)\,dx$$

$$= 4\pi\left[\frac{3}{2}x + 2\cos x - \frac{1}{4}\sin 2x\right]_0^{\pi/2}$$

$$= 4\pi\left(\frac{3\pi}{4} - 2\right) = \pi(3\pi - 8)$$

29. 

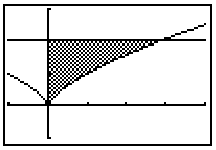
$[-1, 3]$ by $[-1.5, 1.5]$

A cross section has radius $r = \sqrt{5}y^2$ and area

$A(y) = \pi r^2 = 5\pi y^4.$

The volume is $\displaystyle\int_{-1}^{1} 5\pi y^4\,dy = \pi\Big[y^5\Big]_{-1}^{1} = 2\pi.$

30.



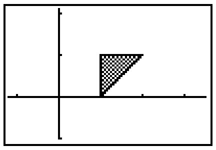
$[-1, 4]$ by $[-1, 3]$

A cross section has radius $r = y^{3/2}$ and area

$A(y) = \pi r^2 = \pi y^3$. The volume is

$\int_0^2 \pi y^3\, dy = \pi\left[\frac{1}{4}y^4\right]_0^2 = 4\pi.$

31.

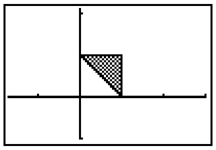


$[-1.2, 3.5]$ by $[-1, 2.1]$

Use washer cross sections. A washer has inner radius $r = 1$, outer radius $R = y + 1$, and area

$A(y) = \pi(R^2 - r^2) = \pi[(y + 1)^2 - 1] = \pi(y^2 + 2y)$. The volume is $\int_0^1 \pi(y^2 + 2y)\, dy = \pi\left[\frac{1}{3}y^3 + y^2\right]_0^1 = \frac{4}{3}\pi.$

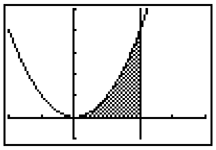
32.



$[-1.7, 3]$ by $[-1, 2.1]$

Use cylindrical shells: a shell has radius x and height x. The volume is $\int_0^1 2\pi(x)(x)\, dx = 2\pi\left[\frac{1}{3}x^3\right]_0^1 = \frac{2}{3}\pi.$

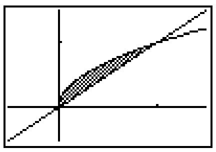
33.



$[-2, 4]$ by $[-1, 5]$

Use cylindrical shells: A shell has radius x and height x^2. The volume is $\int_0^2 2\pi(x)(x^2)\, dx = 2\pi\left[\frac{1}{4}x^4\right]_0^2 = 8\pi.$

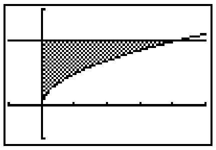
34.



$[-0.5, 1.5]$ by $[-0.5, 1.5]$

The curves intersect at $x = 0$ and $x = 1$. Use cylindrical shells: a shell has radius x and height $\sqrt{x} - x$. The volume is $\int_0^1 2\pi(x)(\sqrt{x} - x)\, dx = 2\pi\left[\frac{2}{5}x^{5/2} - \frac{1}{3}x^3\right]_0^1 = \frac{2\pi}{15}.$

35.



$[-1, 5]$ by $[-1, 3]$

The curved and horizontal line intersect at (4, 2).

(a) Use washer cross sections: a washer has inner radius $r = \sqrt{x}$, outer radius $R = 2$, and area

$A(x) = \pi(R^2 - r^2) = \pi(4 - x)$. The volume is

$\int_0^4 \pi(4 - x)\, dx = \pi\left[4x - \frac{1}{2}x^2\right]_0^4 = 8\pi$

(b) A cross section has radius $r = y^2$ and area

$A(y) = \pi r^2 = \pi y^4$.

The volume is $\int_0^2 \pi y^4\, dy = \pi\left[\frac{1}{5}y^5\right]_0^2 = \frac{32\pi}{5}.$

(c) A cross section has radius $r = 2 - \sqrt{x}$ and area

$A(x) = \pi r^2 = \pi(2 - \sqrt{x})^2 = \pi(4 - 4\sqrt{x} + x).$

The volume is

$\int_0^4 \pi(4 - 4\sqrt{x} + x)\, dx = \pi\left[4x - \frac{8}{3}x^{3/2} + \frac{1}{2}x^2\right]_0^4 = \frac{8\pi}{3}.$

(d) Use washer cross sections: a washer has inner radius

$r = 4 - y^2$, outer radius $R = 4$, and area

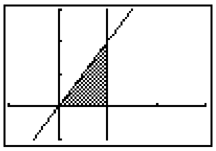
$A(y) = \pi(R^2 - r^2) = \pi[16 - (4 - y^2)^2]$

$= \pi(8y^2 - y^4).$

The volume is

$\int_0^2 \pi(8y^2 - y^4)\, dy = \pi\left[\frac{8}{3}y^3 - \frac{1}{5}y^5\right]_0^2 = \frac{224\pi}{15}$

36.



$[-1, 3]$ by $[-1, 3]$

The slanted and vertical lines intersect at (1, 2)

(a) The solid is a right circular cone of radius 1 and height 2. The volume is

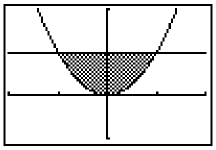
$\frac{1}{3}Bh = \frac{1}{3}(\pi r^2)h = \frac{1}{3}(\pi 1^2)2 = \frac{2}{3}\pi.$

(b) Use cylindrical shells: a shell has radius $2 - x$ and height $2x$. The volume is

$\int_0^1 2\pi(2 - x)(2x)\, dx = 4\pi\int_0^1 (2x - x^2)\, dx$

$= 4\pi\left[x^2 - \frac{1}{3}x^3\right]_0^1 = \frac{8\pi}{3}.$

37.



[−2, 2] by [−1, 2]

The curves intersect at $(\pm 1, 1)$.

(a) A cross section has radius $r = 1 - x^2$ and area

$A(x) = \pi r^2 = \pi(1 - x^2)^2 = \pi(1 - 2x^2 + x^4)$.

The volume is

$$\int_{-1}^{1} \pi(1 - 2x^2 + x^4)\,dx = \pi\left[x - \frac{2}{3}x^3 + \frac{1}{5}x^5\right]_{-1}^{1} = \frac{16\pi}{15}.$$

(b) Use cylindrical shells: a shell has radius $2 - y$ and height $2\sqrt{y}$. The volume is

$$\int_0^1 2\pi(2 - y)(2\sqrt{y})\,dy = 4\pi\int_0^1 (2\sqrt{y} - y^{3/2})\,dy = 4\pi\left[\frac{4}{3}y^{3/2} - \frac{2}{5}y^{5/2}\right]_0^1 = \frac{56\pi}{15}.$$

(c) Use cylindrical shells: a shell has radius $y + 1$ and height $2\sqrt{y}$. The volume is

$$\int_0^1 2\pi(y + 1)(2\sqrt{y})\,dy = 4\pi\int_0^1 (y^{3/2} + \sqrt{y})\,dy = 4\pi\left[\frac{2}{5}y^{5/2} + \frac{2}{3}y^{3/2}\right]_0^1 = \frac{64\pi}{15}.$$

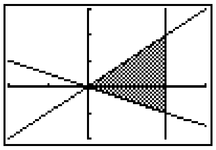
38. **(a)** A cross section has radius $r = h\left(1 - \frac{x}{b}\right)$ and area $A(x) = \pi r^2 = \pi h^2\left(1 - \frac{x}{b}\right)^2$. The volume is

$$\int_0^b \pi h^2\left(1 - \frac{x}{b}\right)^2 dx = \pi h^2\left[-\frac{b}{3}\left(1 - \frac{x}{b}\right)^3\right]_0^b = \frac{\pi}{3}bh^2.$$

(b) Use cylindrical shells: a shell has radius x and height $h\left(1 - \frac{x}{b}\right)$. The volume is

$$\int_0^b 2\pi(x)h\left(1 - \frac{x}{b}\right) dx = 2\pi h\int_0^b \left(x - \frac{x^2}{b}\right) dx = 2\pi h\left[\frac{1}{2}x^2 - \frac{x^3}{3b}\right]_0^b = \frac{\pi}{3}b^2h.$$

39.

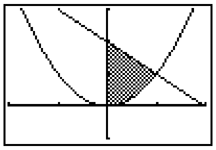


[−2, 3] by [−2, 3]

A shell has radius x and height $x - \left(-\frac{x}{2}\right) = \frac{3}{2}x$.

The volume is $\int_0^2 2\pi(x)\left(\frac{3}{2}x\right) dx = \pi\left[x^3\right]_0^2 = 8\pi$.

40.

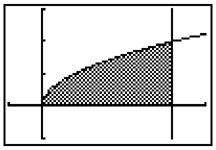


[−2, 2] by [−1, 3]

$x^2 = 2 - x$ at $x = 1$. A shell has radius x and height $2 - x - x^2$. The volume is

$$\int_0^1 2\pi(x)(2 - x - x^2)\,dx = 2\pi\left[x^2 - \frac{1}{3}x^3 - \frac{1}{4}x^4\right]_0^1 = \frac{5\pi}{6}.$$

41.

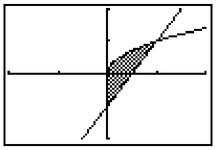


[−1, 5] by [−1, 3]

A shell has radius x and height $\sqrt{x}$. The volume is

$$\int_0^4 2\pi(x)(\sqrt{x})\,dx = 2\pi\left[\frac{2}{5}x^{5/2}\right]_0^4 = \frac{128\pi}{5}.$$

42.



[−2, 2] by [−2, 2]

The functions intersect where $2x - 1 = \sqrt{x}$, i.e., at $x = 1$.

A shell has radius x and height $\sqrt{x} - (2x - 1) = \sqrt{x} - 2x + 1$. The volume is

$$\int_0^1 2\pi(x)(\sqrt{x} - 2x + 1)\,dx = 2\pi\int_0^1 (x^{3/2} - 2x^2 + x)\,dx = 2\pi\left[\frac{2}{5}x^{5/2} - \frac{2}{3}x^3 + \frac{1}{2}x^2\right]_0^1 = \frac{7\pi}{15}.$$

43. A shell has height $12(y^2 - y^3)$.

(a) A shell has radius y. The volume is

$$\int_0^1 2\pi(y)12(y^2 - y^3)\,dy = 24\pi\int_0^1 (y^3 - y^4)\,dy = 24\pi\left[\frac{1}{4}y^4 - \frac{1}{5}y^5\right]_0^1 = \frac{6\pi}{5}.$$

(b) A shell has radius $1 - y$. The volume is

$$\int_0^1 2\pi(1 - y)12(y^2 - y^3)\,dy = 24\pi\int_0^1 (y^4 - 2y^3 + y^2)\,dy = 24\pi\left[\frac{1}{5}y^5 - \frac{1}{2}y^4 + \frac{1}{3}y^3\right]_0^1 = \frac{4\pi}{5}.$$

43. continued

(c) A shell has radius $\frac{8}{5} - y$. The volume is

$$\int_0^1 2\pi\left(\frac{8}{5} - y\right)12(y^2 - y^3)\,dy$$
$$= 24\pi\int_0^1\left(y^4 - \frac{13}{5}y^3 + \frac{8}{5}y^2\right)dy$$
$$= 24\pi\left[\frac{1}{5}y^5 - \frac{13}{20}y^4 + \frac{8}{15}y^3\right]_0^1 = 2\pi.$$

(d) A shell has radius $y + \frac{2}{5}$. The volume is

$$\int_0^1 2\pi\left(y + \frac{2}{5}\right)12(y^2 - y^3)\,dy$$
$$= 24\pi\int_0^1\left(-y^4 + \frac{3}{5}y^3 + \frac{2}{5}y^2\right)dx$$
$$= 24\pi\left[-\frac{1}{5}y^5 + \frac{3}{20}y^4 + \frac{2}{15}y^3\right]_0^1 = 2\pi.$$

44. A shell has height $\frac{y^2}{2} - \left(\frac{y^4}{4} - \frac{y^2}{2}\right) = y^2 - \frac{y^4}{4}.$

(a) A shell has radius y. The volume is

$$\int_0^2 2\pi(y)\left(y^2 - \frac{y^4}{4}\right)dy = 2\pi\left[\frac{1}{4}y^4 - \frac{1}{24}y^6\right]_0^2 = \frac{8\pi}{3}.$$

(b) A shell has radius $2 - y$. The volume is

$$\int_0^2 2\pi(2 - y)\left(y^2 - \frac{y^4}{4}\right)dy$$
$$= 2\pi\int_0^2\left(\frac{y^5}{4} - \frac{y^4}{2} - y^3 + 2y^2\right)dy$$
$$= 2\pi\left[\frac{1}{24}y^6 - \frac{1}{10}y^5 - \frac{1}{4}y^4 + \frac{2}{3}y^3\right]_0^2 = \frac{8\pi}{5}.$$

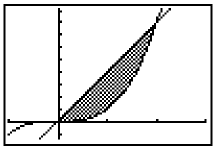
(c) A shell has radius $5 - y$. The volume is

$$\int_0^2 2\pi(5 - y)\left(y^2 - \frac{y^4}{4}\right)dy$$
$$= 2\pi\int_0^2\left(\frac{y^5}{4} - \frac{5y^4}{4} - y^3 + 5y^2\right)dy$$
$$= 2\pi\left[\frac{1}{24}y^6 - \frac{1}{4}y^5 - \frac{1}{4}y^4 + \frac{5}{3}y^3\right]_0^2 = 8\pi.$$

(d) A shell has radius $y + \frac{5}{8}$. The volume is

$$\int_0^2 2\pi\left(y + \frac{5}{8}\right)\left(y^2 - \frac{y^4}{4}\right)dy$$
$$= 2\pi\int_0^2\left(-\frac{y^5}{4} - \frac{5y^4}{32} + y^3 + \frac{5y^2}{8}\right)dy$$
$$= 2\pi\left[-\frac{1}{24}y^6 - \frac{1}{32}y^5 + \frac{1}{4}y^4 + \frac{5}{24}y^3\right]_0^2 = 4\pi.$$

45.



$[-1, 3]$ by $[-1.4, 9.1]$

The functions intersect at (2, 8).

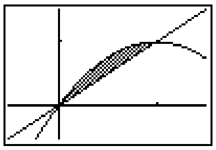
(a) Use washer cross sections: a washer has inner radius $r = x^3$, outer radius $R = 4x$, and area $A(x) = \pi(R^2 - r^2) = \pi(16x^2 - x^6)$. The volume is

$$\int_0^2 \pi(16x^2 - x^6)\,dx = \pi\left[\frac{16}{3}x^3 - \frac{1}{7}x^7\right]_0^2 = \frac{512\pi}{21}.$$

(b) Use cylindrical shells: a shell has a radius $8 - y$ and height $y^{1/3} - \frac{y}{4}$. The volume is

$$\int_0^8 2\pi(8 - y)\left(y^{1/3} - \frac{y}{4}\right)dy$$
$$= 2\pi\int_0^8\left(8y^{1/3} - 2y - y^{4/3} + \frac{y^2}{4}\right)dy$$
$$= 2\pi\left[6y^{4/3} - y^2 - \frac{3}{7}y^{7/3} + \frac{1}{12}y^3\right]_0^8 = \frac{832\pi}{21}.$$

46.



$[-0.5, 1.5]$ by $[-0.5, 1.5]$

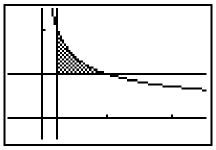
The functions intersect at (0, 0) and (1, 1).

(a) Use cylindrical shells: a shell has radius x and height $2x - x^2 - x = x - x^2$. The volume is

$$\int_0^1 2\pi(x)(x - x^2)\,dx = 2\pi\left[\frac{1}{3}x^3 - \frac{1}{4}x^4\right]_0^1 = \frac{\pi}{6}.$$

(b) Use cylindrical shells: a shell has radius $1 - x$ and height $2x - x^2 - x = x - x^2$. The volume is

$$\int_0^1 2\pi(1 - x)(x - x^2)\,dx = 2\pi\int_0^1(x^3 - 2x^2 + x)\,dx$$
$$= 2\pi\left[\frac{1}{4}x^4 - \frac{2}{3}x^3 + \frac{1}{2}x^2\right]_0^1$$
$$= \frac{\pi}{6}.$$

47. 

[−0.5, 2.5] by [−0.5, 2.5]

The intersection points are $\left(\frac{1}{4}, 1\right)$, $\left(\frac{1}{4}, 2\right)$, and (1, 1).

(a) A washer has inner radius $r = \frac{1}{4}$, outer radius $R = \frac{1}{y^2}$, and area $\pi(R^2 - r^2) = \pi\left(\frac{1}{y^4} - \frac{1}{16}\right)$. The volume is $\int_1^2 \pi\left(\frac{1}{y^4} - \frac{1}{16}\right) dy = \pi\left[-\frac{1}{3y^3} - \frac{1}{16}y\right]_1^2 = \frac{11\pi}{48}$.

(b) A shell has radius x and height $\frac{1}{\sqrt{x}} - 1$. The volume is $\int_{1/4}^1 2\pi(x)\left(\frac{1}{\sqrt{x}} - 1\right) dx = 2\pi\left[\frac{2}{3}x^{3/2} - \frac{1}{2}x^2\right]_{1/4}^1 = \frac{11\pi}{48}$.

48. **(a)** For $0 < x \le \pi$, $x f(x) = \frac{x(\sin x)}{x} = \sin x$.

For $x = 0$, $x f(x) = 0 \cdot 1 = \sin 0 = \sin x$. So

$x f(x) = \sin x$ for $0 \le x \le \pi$.

(b) Use cylindrical shells: a shell has radius x and height y. The volume is $\int_0^\pi 2\pi xy\, dx$, which from part (a) is $\int_0^\pi 2\pi \sin x\, dx = 2\pi\left[-\cos x\right]_0^\pi = 4\pi$.

49. **(a)** A cross section has radius $r = \frac{x}{12}\sqrt{36 - x^2}$ and area $A(x) = \pi r^2 = \frac{\pi}{144}(36x^2 - x^4)$. The volume is $\int_0^6 \frac{\pi}{144}(36x^2 - x^4)\, dx = \frac{\pi}{144}\left[12x^3 - \frac{1}{5}x^5\right]_0^6 = \frac{36\pi}{5}$ cm^3.

(b) $\left(\frac{36\pi}{5}\text{ cm}^3\right)(8.5\text{ g/cm}^3) \approx 192.3$ g

50. A cross section has radius $r = |c - \sin x|$ and area

$A(x) = \pi r^2 = \pi(c - \sin x)^2 = \pi(c^2 - 2c\sin x + \sin^2 x)$.

The volume is

$$\int_0^\pi \pi(c^2 - 2c\sin x + \sin^2 x)\, dx$$
$$= \pi\left[c^2x + 2c\cos x + \frac{1}{2}x - \frac{1}{4}\sin 2x\right]_0^\pi$$
$$= \pi\left[\left(c^2\pi - 2c + \frac{1}{2}\pi\right) - 2c\right]$$
$$= \pi^2c^2 - 4\pi c + \frac{\pi^2}{2}.$$

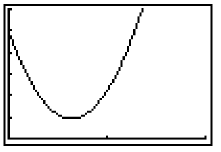
(a) Solve

$$\frac{d}{dc}\left[\pi^2c^2 - 4\pi c + \frac{\pi^2}{2}\right] = 0$$
$$2\pi^2c - 4\pi = 0$$
$$\pi c - 2 = 0$$
$$c = \frac{2}{\pi}$$

This value of c gives a minimum for V because $\frac{d^2V}{dc^2} = 2\pi^2 > 0$.

Then the volume is $\pi^2\left(\frac{2}{\pi}\right)^2 - 4\pi\left(\frac{2}{\pi}\right) + \frac{\pi^2}{2} = \frac{\pi^2}{2} - 4$

(b) Since the derivative with respect to c is not zero anywhere else besides $c = \frac{2}{\pi}$, the maximum must occur at $c = 0$ or $c = 1$. The volume for $c = 0$ is $\frac{\pi^2}{2} \approx 4.935$, and for $c = 1$ it is $\frac{\pi(3\pi - 8)}{2} \approx 2.238$. $c = 0$ maximizes the volume.

(c) 

[0, 2] by [0, 6]

The volume gets large without limit. This makes sense, since the curve is sweeping out space in an ever-increasing radius.

51. **(a)** Using $d = \frac{C}{\pi}$, and $A = \pi\left(\frac{d}{2}\right)^2 = \frac{C^2}{4\pi}$ yields the following areas (in square inches, rounded to the nearest tenth): 2.3, 1.6, 1.5, 2.1, 3.2, 4.8, 7.0, 9.3, 10.7, 10.7, 9.3, 6.4, 3.2.

(b) If $C(y)$ is the circumference as a function of y, then the area of a cross section is

$A(y) = \pi\left(\frac{C(y)/\pi}{2}\right)^2 = \frac{C^2(y)}{4\pi}$,

and the volume is $\frac{1}{4\pi}\int_0^6 C^2(y)\, dy$.

(c) $\frac{1}{4\pi}\int_0^6 A(y)\, dy = \frac{1}{4\pi}\int_0^6 C^2(y)\, dy$

$$\approx \frac{1}{4\pi}\left(\frac{6-0}{24}\right)[5.4^2 + 2(4.5^2 + 4.4^2 + 5.1^2 + 6.3^2 + 7.8^2 + 9.4^2 + 10.8^2 + 11.6^2 + 11.6^2 + 10.8^2 + 9.0^2) + 6.3^2] \approx 34.7 \text{ in.}^3$$

52. (a) A cross section has radius $r = \sqrt{2y}$ and area $\pi r^2 = 2\pi y$. The volume is $\int_0^5 2\pi y\, dy = \pi\left[y^2\right]_0^5 = 25\pi$.

(b) $V(h) = \int A(h)\, dh$, so $\frac{dV}{dh} = A(h)$.

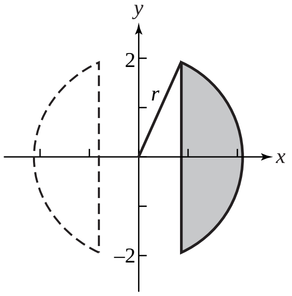
$\therefore \frac{dV}{dt} = \frac{dV}{dh} \cdot \frac{dh}{dt} = A(h) \cdot \frac{dh}{dt}$,

so $\frac{dh}{dt} = \frac{1}{A(h)} \cdot \frac{dV}{dt}$

For $h = 4$, the area is $2\pi(4) = 8\pi$,

so $\frac{dh}{dt} = \frac{1}{8\pi} \cdot 3\ \frac{\text{units}^3}{\text{sec}} = \frac{3}{8\pi} \cdot \frac{\text{units}^3}{\text{sec}}$.

53. (a)



The remaining solid is that swept out by the shaded region in revolution. Use cylindrical shells: a shell has radius x and height $2\sqrt{r^2 - x^2}$. The volume is

$$\int_{\sqrt{r^2 - 2^2}}^{r} 2\pi(x)(2\sqrt{r^2 - x^2})\, dx$$

$$= 2\pi\left[-\frac{2}{3}(r^2 - x^2)^{3/2}\right]_{\sqrt{r^2 - 4}}^{r}$$

$$= -\frac{4}{3}\pi(-8) = \frac{32\pi}{3}.$$

(b) The answer is independent of r.

54. Partition the appropriate interval in the axis of revolution and measure the radius $r(x)$ of the shadow region at these points. Then use an approximation such as the trapezoidal rule to estimate the integral $\int_a^b \pi r^2(x)\, dx$.

55. Solve $ax - x^2 = 0$: This is true at $x = 0$ and $x = a$. For revolution about the x-axis, a cross section has radius $r = ax - x^2$ and area

$A(x) = \pi r^2 = \pi(ax - x^2)^2 = \pi(a^2x^2 - 2ax^3 + x^4)$.

The volume is

$$\int_0^a \pi(a^2x^2 - 2ax^3 + x^4)\, dx = \pi\left[\frac{1}{3}a^2x^3 - \frac{1}{2}ax^4 + \frac{1}{5}x^5\right]_0^a = \frac{1}{30}\pi a^5.$$

For revolution about the y-axis, a cylindrical shell has radius x and height $ax - x^2$. The volume is

$$\int_0^a 2\pi(x)(ax - x^2)\, dx = 2\pi\left[\frac{1}{3}ax^3 - \frac{1}{4}x^4\right]_0^a = \frac{1}{6}\pi a^4.$$

Setting the two volumes equal,

$\frac{1}{30}\pi a^5 = \frac{1}{6}\pi a^4$ yields $\frac{1}{30}a = \frac{1}{6}$, so $a = 5$.

56. The slant height Δs of a tiny horizontal slice can be written as $\Delta s = \sqrt{\Delta x^2 + \Delta y^2} = \sqrt{1 + (g'(y))^2}\Delta y$. So the surface area is approximated by the Riemann sum $\sum_{k=1}^{n} 2\pi g(y_k)\sqrt{1 + (g'(y))^2}\ \Delta y$. The limit of that is the integral.

57. $g'(y) = \frac{dx}{dy} = \frac{1}{2\sqrt{y}}$, and

$$\int_0^2 2\pi\sqrt{y}\sqrt{1 + \left(\frac{1}{2\sqrt{y}}\right)^2}\, dy = \int_0^2 \pi\sqrt{4y + 1}\, dy = \left[\frac{\pi}{6}(4y + 1)^{3/2}\right]_0^2 = \frac{13\pi}{3} \approx 13.614$$

58. $g'(y) = \frac{dx}{dy} = y^2$, and

$$\int_0^1 2\pi\left(\frac{y^3}{3}\right)\sqrt{1 + (y^2)^2}\, dy = \frac{2}{3}\pi\left[\frac{1}{6}(1 + y^4)^{3/2}\right]_0^1 = \frac{\pi}{9}(2\sqrt{2} - 1) \approx 0.638.$$

59. $g'(y) = \frac{dx}{dy} = \frac{1}{2}y^{-1/2}$, and

$$\int_1^3 2\pi\left[y^{1/2} - \left(\frac{1}{3}\right)^{3/2}\right]\sqrt{1 + \left[\frac{1}{2}y^{-1/2}\right]^2}\, dy$$

$$= 2\pi\int_1^3\left[y^{1/2} - \left(\frac{1}{3}\right)^{3/2}\right]\sqrt{1 + \frac{1}{4y}}\, dy.$$

Using NINT, this evaluates to ≈ 16.110

60. $g'(y) = \frac{dx}{dy} = \frac{1}{\sqrt{2y-1}}$, and

$$\int_{5/8}^{1} 2\pi\sqrt{2y-1}\sqrt{1+\left(\frac{1}{\sqrt{2y-1}}\right)^2}\,dy$$

$$= 2\pi\int_{5/8}^{1}\sqrt{2y}\,dy$$

$$= 2\sqrt{2}\pi\left[\frac{2}{3}y^{3/2}\right]_{5/8}^{1}$$

$$= \frac{4\sqrt{2}}{3}\pi\left(1-\frac{5}{16}\sqrt{\frac{5}{2}}\right) \approx 2.997.$$

61. $f'(x) = \frac{dy}{dx} = 2x$, and

$\int_0^2 2\pi x^2\sqrt{1+(2x)^2}\,dx = \int_0^2 2\pi x^2\sqrt{1+4x^2}\,dx$ evaluates, using NINT, to ≈ 53.226.

62. $f'(x) = \frac{dy}{dx} = 3-2x$, and

$\int_0^3 2\pi(3x-x^2)\sqrt{1+(3-2x)^2}\,dx$ evaluates, using NINT, to ≈ 44.877.

63. $f'(x) = \frac{dy}{dx} = \frac{1-x}{\sqrt{2x-x^2}}$, and

$$\int_{0.5}^{1.5} 2\pi\sqrt{2x-x^2}\sqrt{1+\left(\frac{1-x}{\sqrt{2x-x^2}}\right)^2}\,dx = 2\pi\int_{0.5}^{1.5}1\,dx$$

$$= 2\pi\Big[x\Big]_{0.5}^{1.5}$$

$$= 2\pi \approx 6.283$$

64. $f'(x) = \frac{dy}{dx} = \frac{1}{2\sqrt{x+1}}$, and

$$\int_1^5 2\pi\sqrt{x+1}\sqrt{1+\left(\frac{1}{2\sqrt{x+1}}\right)^2}\,dx$$

$$= 2\pi\int_1^5\sqrt{x+\frac{5}{4}}\,dx$$

$$= 2\pi\left[\frac{2}{3}\left(x+\frac{5}{4}\right)^{3/2}\right]_1^5$$

$$= \frac{4\pi}{3}\left[\left(\frac{25}{4}\right)^{3/2}-\left(\frac{9}{4}\right)^{3/2}\right] = \frac{49\pi}{3} \approx 51.313$$

65. Hemisphere cross sectional area: $\pi(\sqrt{R^2-h^2})^2 = A_1$.
Right circular cylinder with cone removed cross sectional area: $\pi R^2 - \pi h^2 = A_2$
Since $A_1 = A_2$, the two volumes are equal by Cavalieri's theorem. Thus, volume of hemisphere
$=$ volume of cylinder $-$ volume of cone
$= \pi R^3 - \frac{1}{3}\pi R^3 = \frac{2}{3}\pi R^3$.

66. Use washer cross sections: a washer has inner radius $r = b - \sqrt{a^2-y^2}$, outer radius $R = b + \sqrt{a^2-y^2}$, and area $\pi(R^2 - r^2)$

$$= \pi[(b+\sqrt{a^2-y^2})^2 - (b-\sqrt{a^2-y^2})^2]$$

$= 4\pi b\sqrt{a^2-y^2}$. The volume is

$$\int_{-a}^{a} 4\pi b\sqrt{a^2-y^2}\,dy = 4\pi b\int_{-a}^{a}\sqrt{a^2-y^2}\,dy$$

$$= 4\pi b\left(\frac{\pi a^2}{2}\right)$$

$$= 2\pi^2a^2b$$

67. **(a)** Put the bottom of the bowl at $(0, -a)$. The area of a horizontal cross section is $\pi(\sqrt{a^2-y^2})^2 = \pi(a^2-y^2)$. The volume for height h is

$$\int_{-a}^{h-a}\pi(a^2-y^2)\,dy = \pi\left[a^2y - \frac{1}{3}y^3\right]_{-a}^{h-a} = \frac{\pi h^2(3a-h)}{3}.$$

(b) For $h = 4$, $y = -1$ and the area of a cross section is $\pi(5^2 - 1^2) = 24\pi$. The rate of rise is $\frac{dh}{dt} = \frac{1}{A}\frac{dV}{dt} = \frac{1}{24\pi}(0.2) = \frac{1}{120\pi}$ m/sec.

68. **(a)** A cross section has radius $r = \sqrt{a^2-x^2}$ and area $A(x) = \pi r^2 = \pi(\sqrt{a^2-x^2})^2 = \pi(a^2-x^2)$. The volume is

$$\int_{-a}^{a}\pi(a^2-x^2)\,dx = \pi\left[a^2x - \frac{1}{3}x^3\right]_{-a}^{a}$$

$$= \pi\left[\left(a^3-\frac{1}{3}a^3\right) - \left(-a^3+\frac{1}{3}a^3\right)\right]$$

$$= \frac{4}{3}\pi a^3.$$

(b) A cross section has radius $x = r\left(1-\frac{y}{h}\right)$ and area $A(y) = \pi x^2 = \pi r^2\left(1-\frac{y}{h}\right)^2 = \pi r^2\left(1-\frac{2y}{h}+\frac{y^2}{h^2}\right)$. The volume is

$$\int_0^h \pi r^2\left(1-\frac{2y}{h}+\frac{y^2}{h^2}\right)dy = \pi r^2\left[y - \frac{y^2}{h} + \frac{y^3}{3h^2}\right]_0^h$$

$$= \frac{1}{3}\pi r^2h.$$

■ Section 7.4 Lengths of Curves
(pp. 395–401)

Quick Review 7.4

1. $\sqrt{1+2x+x^2} = \sqrt{(1+x)^2}$, which, since $x \ge -1$, equals $1 + x$ or $x + 1$.

2. $\sqrt{1 - x + \frac{x^2}{4}} = \sqrt{\left(1 - \frac{x}{2}\right)^2}$, which, since $x \le 2$, equals $1 - \frac{x}{2}$ or $\frac{2-x}{2}$.

3. $\sqrt{1 + (\tan x)^2} = \sqrt{(\sec x)^2}$, which, since $0 \le x < \frac{\pi}{2}$, equals $\sec x$.

4. $\sqrt{1 + \left(\frac{x}{4} - \frac{1}{x}\right)^2} = \sqrt{\frac{1}{2} + \frac{1}{16}x^2 + \frac{1}{x^2}} = \frac{1}{4}\sqrt{\frac{(x^2+4)^2}{x^2}}$ which, since $x > 0$, equals $\frac{x^2+4}{4x}$.

5. $\sqrt{1 + \cos 2x} = \sqrt{2\cos^2 x}$, which, since $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$, equals $\sqrt{2}\cos x$.

6. $f(x)$ has a corner at $x = 4$.

7. $\frac{d}{dx}(5x^{2/3}) = \frac{10}{3\sqrt[3]{x}}$ is undefined at $x = 0$. $f(x)$ has a cusp there.

8. $\frac{d}{dx}(\sqrt[5]{x+3}) = \frac{1}{5(x+3)^{4/5}}$ is undefined for $x = -3$. $f(x)$ has a vertical tangent there.

9. $\sqrt{x^2 - 4x + 4} = |x - 2|$ has a corner at $x = 2$.

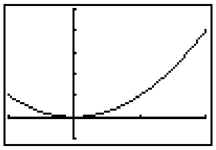
10. $\frac{d}{dx}\left(1 + \sqrt[3]{\sin x}\right) = \frac{\cos x}{3(\sin x)^{2/3}}$ is undefined for $x = k\pi$, where k is any integer. $f(x)$ has vertical tangents at these values of x.

Section 7.4 Exercises

1. (a) $y' = 2x$, so

$$\text{Length} = \int_{-1}^{2}\sqrt{1 + (2x)^2}\,dx = \int_{-1}^{2}\sqrt{1 + 4x^2}\,dx.$$

(b)

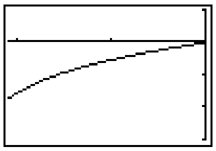


$[-1, 2]$ by $[-1, 5]$

(c) Length ≈ 6.126

2. (a) $y' = \sec^2 x$, so Length $= \int_{-\pi/3}^{0}\sqrt{1 + \sec^4 x}\,dx$.

(b)

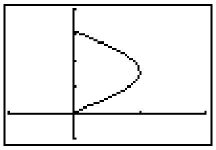


$\left[-\frac{\pi}{3}, 0\right]$ by $[-3, 1]$

(c) Length ≈ 2.057

3. (a) $x' = \cos y$, so Length $= \int_{0}^{\pi}\sqrt{1 + \cos^2 y}\,dy$.

(b)



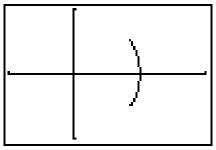
$[-1, 2]$ by $[-1, 4]$

(c) Length ≈ 3.820

4. (a) $x' = -y(1 - y^2)^{-1/2}$, so

$$\text{Length} = \int_{-1/2}^{1/2}\sqrt{1 + \frac{y^2}{1 - y^2}}\,dy.$$

(b)



$[-1, 2]$ by $[-1, 1]$

(c) Length ≈ 1.047

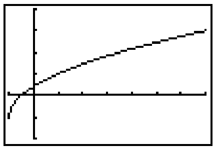
5. (a) $y^2 + 2y = 2x + 1$, so

$y^2 + 2y + 1 = (y + 1)^2 = 2x + 2$, and

$y = \sqrt{2x + 2} - 1$. Then $y' = \frac{1}{\sqrt{2x + 2}}$, and

$$\text{Length} = \int_{-1}^{7}\sqrt{1 + \frac{1}{2x + 2}}\,dx.$$

(b)



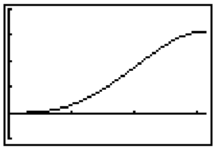
$[-1, 7]$ by $[-2, 4]$

(c) Length ≈ 9.294

6. (a) $y' = \cos x + x\sin x - \cos x = x\sin x$, so

$$\text{Length} = \int_{0}^{\pi}\sqrt{1 + x^2\sin^2 x}\,dx.$$

(b)

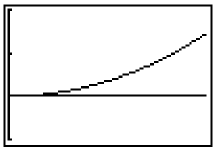


$[0, \pi]$ by $[-1, 4]$

(c) Length ≈ 4.698

7. (a) $y' = \tan x$, so Length $= \int_{0}^{\pi/6}\sqrt{1 + \tan^2 x}\,dx$.

(b) $y = \int \tan x\,dx = \ln(|\sec x|)$



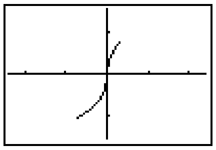
$\left[0, \frac{\pi}{6}\right]$ by $[-0.1, 0.2]$

(c) Length ≈ 0.549

8. (a) $x' = \sqrt{\sec^2 y - 1}$, so Length $= \int_{-\pi/3}^{\pi/4} \sec y \, dy$.

(b) $x' = \sqrt{\sec^2 y - 1} = |\tan y|$,

$$\text{so } x = \begin{cases} \ln(\cos y) & -\frac{\pi}{3} \le y \le 0 \\ -\ln(\cos y) & 0 < y \le \frac{\pi}{4} \end{cases}$$



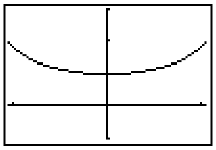
$[-2.4, 2.4]$ by $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

(c) Length ≈ 2.198

9. (a) $y' = \sec x \tan x$, so

Length $= \int_{-\pi/3}^{\pi/3} \sqrt{1 + \sec^2 x \tan^2 x} \, dx$.

(b)

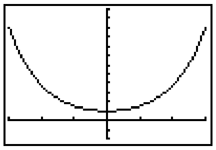


$\left[-\frac{\pi}{3}, \frac{\pi}{3}\right]$ by $[-1, 3]$

(c) Length ≈ 3.139

10. $y' = \frac{(e^x - e^{-x})}{2}$, so Length $= \int_{-3}^{3} \sqrt{1 + \left(\frac{e^x - e^{-x}}{2}\right)^2} \, dx$.

(b)



$[-3, 3]$ by $[-2, 12]$

(c) Length ≈ 20.036

11. $y' = \frac{1}{2}(x^2 + 2)^{1/2}(2x) = x\sqrt{x^2 + 2}$, so the length is

$$\int_0^3 \sqrt{1 + (x\sqrt{x^2 + 2})^2} \, dx = \int_0^3 \sqrt{x^4 + 2x^2 + 1} \, dx$$
$$= \int_0^3 (x^2 + 1) \, dx = \left[\frac{1}{3}x^3 + x\right]_0^3 = 12.$$

12. $y' = \frac{3}{2}\sqrt{x}$, so the length is

$$\int_0^4 \sqrt{1 + \left(\frac{3}{2}\sqrt{x}\right)^2} \, dx = \int_0^4 \sqrt{1 + \frac{9x}{4}} \, dx$$
$$= \left[\frac{8}{27}\left(1 + \frac{9x}{4}\right)^{3/2}\right]_0^4$$
$$= \frac{80\sqrt{10} - 8}{27}.$$

13. $x' = y^2 - \frac{1}{4y^2}$, so the length is $\int_1^3 \sqrt{1 + \left(y^2 - \frac{1}{4y^2}\right)^2} \, dy$

$$= \int_1^3 \sqrt{\left(y^2 + \frac{1}{4y^2}\right)^2} \, dy = \left[\frac{1}{3}y^3 - \frac{1}{4y}\right]_1^3 = \frac{53}{6}.$$

14. $x' = y^3 - \frac{1}{4y^3}$, so the length is

$$\int_1^2 \sqrt{1 + \left(y^3 - \frac{1}{4y^3}\right)^2} \, dy = \int_1^2 \sqrt{\left(y^3 + \frac{1}{4y^3}\right)^2} \, dy$$
$$= \left[\frac{1}{4}y^4 - \frac{1}{8y^2}\right]_1^2$$
$$= \frac{123}{32}.$$

15. $x' = \frac{y^2}{2} - \frac{1}{2y^2}$, so the length is

$$\int_1^2 \sqrt{1 + \left(\frac{y^2}{2} - \frac{1}{2y^2}\right)^2} \, dy = \int_1^2 \sqrt{\left(\frac{y^2}{2} + \frac{1}{2y^2}\right)^2} \, dy$$
$$= \left[\frac{1}{6}y^3 - \frac{1}{2y}\right]_1^2 = \frac{17}{12}.$$

16. $y' = x^2 + 2x + 1 - \frac{4}{(4x + 4)^2} = (x + 1)^2 - \frac{1}{4(x + 1)^2}$

so the length is

$$\int_0^2 \sqrt{1 + \left((x + 1)^2 - \frac{1}{4(x + 1)^2}\right)^2} \, dx$$
$$= \int_0^2 \sqrt{\left((x + 1)^2 + \frac{1}{4(x + 1)^2}\right)^2} \, dx$$
$$= \left[\frac{1}{3}(x + 1)^3 - \frac{1}{4(x + 1)}\right]_0^2 = \frac{53}{6}.$$

17. $x' = \sqrt{\sec^4 y - 1}$, so the length is

$$\int_{-\pi/4}^{\pi/4} \sqrt{1 + (\sec^4 y - 1)}\, dy = \int_{-\pi/4}^{\pi/4} \sec^2 y\, dy = \Big[\tan y\Big]_{-\pi/4}^{\pi/4} = 2.$$

18. $y' = \sqrt{3x^4 - 1}$, so the length is

$$\int_{-2}^{-1} \sqrt{1 + (3x^4 - 1)}\, dx = \int_{-2}^{-1} \sqrt{3}x^2\, dx = \sqrt{3}\left[\frac{1}{3}x^3\right]_{-2}^{-1} = \frac{7\sqrt{3}}{3}.$$

19. (a) $\left(\frac{dy}{dx}\right)^2$ corresponds to $\frac{1}{4x}$ here, so take $\frac{dy}{dx}$ as $\frac{1}{2\sqrt{x}}$. Then $y = \sqrt{x} + C$, and, since (1, 1) lies on the curve, $C = 0$. So $y = \sqrt{x}$.

(b) Only one. We know the derivative of the function and the value of the function at one value of x.

20. (a) $\left(\frac{dx}{dy}\right)^2$ corresponds to $\frac{1}{y^4}$ here, so take $\frac{dx}{dy}$ as $\frac{1}{y^2}$. Then $x = -\frac{1}{y} + C$ and, since (0, 1) lies on the curve, $C = 1$. So $y = \frac{1}{1 - x}$.

(b) Only one. We know the derivative of the function and the value of the function at one value of x.

21. $y' = \sqrt{\cos 2x}$, so the length is

$$\int_0^{\pi/4} \sqrt{1 + \cos 2x}\, dx = \int_0^{\pi/4} \sqrt{2\cos^2 x}\, dx = \sqrt{2}\Big[\sin x\Big]_0^{\pi/4} = 1.$$

22. $y' = -(1 - x^{2/3})^{1/2}x^{-1/3}$, so the length is

$$8\int_{\sqrt{2}/4}^{1} \sqrt{1 + (1 - x^{2/3})x^{-2/3}}\, dx$$
$$= 8\int_{\sqrt{2}/4}^{1} \sqrt{x^{-2/3}}\, dx$$
$$= 8\int_{\sqrt{2}/4}^{1} x^{-1/3}\, dx$$
$$= 8\left[\frac{3}{2}x^{2/3}\right]_{\sqrt{2}/4}^{1}$$
$$= 8\left[\frac{3}{2} - \frac{3}{2}\left(\frac{1}{2}\right)\right] = 6.$$

23. Find the length of the curve $y = \sin \frac{3\pi}{20}x$ for $0 \le x \le 20$.

$y' = \frac{3\pi}{20}\cos\frac{3\pi}{20}x$, so the length is $\int_0^{20} \sqrt{1 + \left(\frac{3\pi}{20}\cos\frac{3\pi}{20}x\right)^2}\, dx$, which evaluates, using NINT, to ≈ 21.07 inches.

24. The area is 300 times the length of the arch.

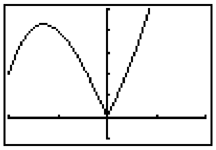
$y' = -\left(\frac{\pi}{2}\right)\sin\left(\frac{\pi x}{50}\right)$, so the length is $\int_{-25}^{25} \sqrt{1 + \left(\frac{\pi}{2}\right)^2 \sin^2\left(\frac{\pi x}{50}\right)}\, dx$, which evaluates, using NINT, to ≈ 73.185. Multiply that by 300, then by \$1.75 to obtain the cost (rounded to the nearest dollar): \$38,422.

25. For track 1: $y_1 = 0$ at $x = \pm 10\sqrt{5} \approx \pm 22.3607$, and $y_1' = \frac{-0.2x}{\sqrt{100 - 0.2x^2}}$. NINT fails to evaluate $\int_{-10\sqrt{5}}^{10\sqrt{5}} \sqrt{1 + (y_1')^2}\, dx$ because of the undefined slope at the limits, so use the track's symmetry, and "back away" from the upper limit a little, and find $2\int_0^{22.36} \sqrt{1 + (y_1')^2}\, dx \approx 52.548$. Then, pretending the last little stretch at each end is a straight line, add $2\sqrt{100 - 0.2(22.36)^2} \approx 0.156$ to get the total length of track 1 as ≈ 52.704. Using a similar strategy, find the length of the *right half* of track 2 to be ≈ 32.274. Now enter $Y_1 = 52.704$ and $Y_2 = 32.274 + \text{NINT}\left(\sqrt{1 + \left(\frac{-0.2t}{\sqrt{150 - 0.2t^2}}\right)^2}, t, x, 0\right)$ and graph in a $[-30, 0]$ by $[0, 60]$ window to see the effect of the x-coordinate of the lane-2 starting position on the length of lane 2. (Be patient!) Solve graphically to find the intersection at $x \approx -19.909$, which leads to starting point coordinates $(-19.909, 8.410)$.

26. $f'(x) = \frac{1}{3}x^{-2/3} + \frac{2}{3}x^{-1/3}$, but NINT fails on $\int_0^2 \sqrt{1 + (f'(x))^2}\, dx$ because of the undefined slope at $x = 0$. So, instead solve for x in terms of y using the quadratic formula. $(x^{1/3})^2 + x^{1/3} - y = 0$, and $x^{1/3} = \frac{-1 \pm \sqrt{1 + 4y}}{2}$. Using the positive values, $x = \frac{1}{8}(\sqrt{1 + 4y} - 1)^3$. Then, $x' = \frac{3}{8}(\sqrt{1 + 4y} - 1)^2\left(\frac{2}{\sqrt{1 + 4y}}\right)$, and $\int_0^{2^{1/3} + 2^{2/3}} \sqrt{1 + (x')^2}\, dy \approx 3.6142$.

27. $f'(x) = \frac{(4x^2+1)-(8x^2-8x)}{(4x^2+1)^2} = -\frac{4x^2-8x-1}{(4x+1)^2}$, so the length is $\int_{-1/2}^{1}\sqrt{1+\left(\frac{4x^2-8x-1}{(4x^2+1)^2}\right)^2}\,dx$ which evaluates, using NINT, to ≈ 2.1089.

28. There is a corner at $x = 0$:



$[-2, 2]$ by $[-1, 5]$

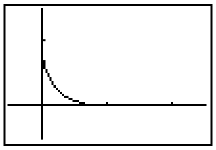
Break the function into two smooth segments:

$$y = \begin{cases} x^3 - 5x & -2 \le x \le 0 \\ x^3 + 5x & 0 < x \le 1 \end{cases} \text{ and}$$

$$y' = \begin{cases} 3x^2 - 5 & -2 \le x < 0 \\ 3x^2 + 5 & 0 < x \le 1 \end{cases}.$$

The length is $\int_{-2}^{1}\sqrt{1+(y')^2}\,dy$
$= \int_{-2}^{0}\sqrt{1+(3x^2-5)^2}\,dx + \int_{0}^{1}\sqrt{1+(3x^2+5)^2}\,dx,$
which evaluates, using NINT for each part, to ≈ 13.132.

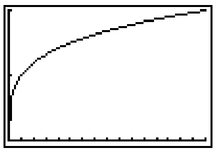
29. $y = (1-\sqrt{x})^2,\ 0 \le x \le 1$



$[-0.5, 2.5]$ by $[-0.5, 1.5]$

$y' = \frac{\sqrt{x}-1}{\sqrt{x}}$, but NINT may fail using y' over the entire interval because y' is undefined at $x = 0$. So, split the curve into two equal segments by solving $\sqrt{x} + \sqrt{y} = 1$ with $y = x$: $x = \frac{1}{4}$. The total length is $2\int_{1/4}^{1}\sqrt{1+\left(\frac{\sqrt{x}-1}{\sqrt{x}}\right)^2}\,dx$, which evaluates, using NINT, to ≈ 1.623.

30.



$[0, 16]$ by $[0, 2]$

$y' = \frac{1}{4}x^{-3/4}$, but NINT may fail using y' over the entire interval, because y' is undefined at $x = 0$. So, use $x = y^4,\ 0 \le y \le 2$: $x' = 4y^3$ and the length is $\int_0^2\sqrt{1+(4y^3)^2}\,dy$, which evaluates, using NINT, to ≈ 16.647.

31. Because the limit of the sum $\sum \Delta x_k$ as the norm of the partition goes to zero will always be the length $(b-a)$ of the interval (a, b).

32. No; the curve can be indefinitely long. Consider, for example, the curve $\frac{1}{3}\sin\left(\frac{1}{x}\right) + 0.5$ for $0 < x < 1$.

33. (a) The fin is the hypotenuse of a right triangle with leg lengths Δx_k and $\left.\frac{df}{dx}\right|_{x=x_{k-1}}\Delta x_k = f'(x_{k-1})\Delta x_k$.

(b) $\lim_{n\to\infty}\sum_{k=1}^{n}\sqrt{(\Delta x_k)^2 + (f'(x_{k-1})\Delta x_k)^2}$
$= \lim_{n\to\infty}\sum_{k=1}^{n}\Delta x_k\sqrt{1+(f'(x_{k-1}))^2}$
$= \int_a^b\sqrt{1+(f'(x))^2}\,dx$

34. Yes. Any curve of the form $y = \pm x + c$, where c is a constant, has constant slope ± 1, so that $\int_0^a\sqrt{1+(y')^2}\,dx = \int_0^a\sqrt{2}\,dx = a\sqrt{2}$.

■ Section 7.5 Applications from Science and Statistics
(pp. 401–411)

Quick Review 7.5

1. (a) $\int_0^1 e^{-x}\,dx = \left[-e^{-x}\right]_0^1 = 1 - \frac{1}{e}$

(b) ≈ 0.632

2. (a) $\int_0^1 e^x\,dx = \left[e^x\right]_0^1 = e - 1$

(b) ≈ 1.718

3. (a) $\int_{\pi/4}^{\pi/2}\sin x\,dx = \left[-\cos x\right]_{\pi/4}^{\pi/2} = \frac{\sqrt{2}}{2}$

(b) ≈ 0.707

4. (a) $\int_0^3(x^2+2)\,dx = \left[\frac{1}{3}x^3 + 2x\right]_0^3 = 15$

(b) 15

5. (a) $\int_1^2\frac{x^2}{x^3+1}\,dx = \left[\frac{1}{3}\ln(x^3+1)\right]_1^2$
$= \frac{1}{3}[\ln 9 - \ln 2]$
$= \frac{1}{3}\ln\left(\frac{9}{2}\right)$

(b) ≈ 0.501

6. $\int_0^7 2\pi(x+2)\sin x\,dx$

7. $\int_0^7(1-x^2)(2\pi x)\,dx$

8. $\int_0^7 \pi \cos^2 x\,dx$

9. $\int_0^7 \pi\left(\frac{y}{2}\right)^2(10 - y)\,dy$

10. $\int_0^7 \frac{\sqrt{3}}{4}\sin^2 x\,dx$

Section 7.5 Exercises

1. $\int_0^5 xe^{-x/3}\,dx = \Big[-3e^{-x/3}(3 + x)\Big]_0^5 = -\frac{24}{e^{5/3}} + 9 \approx 4.4670$ J

2. $\int_0^3 x\sin\left(\frac{\pi x}{4}\right)dx = \frac{4}{\pi}\left[\frac{4}{\pi}\sin\left(\frac{\pi x}{4}\right) - x\cos\left(\frac{\pi x}{4}\right)\right]_0^3$

$= \frac{4\sqrt{2}}{\pi}\left(\frac{2}{\pi} + \frac{3}{2}\right)$

≈ 3.8473 J

3. $\int_0^3 x\sqrt{9 - x^2}\,dx = \left[-\frac{1}{3}(9 - x^2)^{3/2}\right]_0^3 = 9$ J

4. $\int_0^{10}(e^{\sin x}\cos x + 2)\,dx = \Big[e^{\sin x} + 2x\Big]_0^{10}$

$= e^{\sin 10} + 19 \approx 19.5804$ J

5. When the bucket is x m off the ground, the water weighs

$F(x) = 490\left(\frac{20 - x}{20}\right) = 490\left(1 - \frac{x}{20}\right) = 490 - 24.5x$ N.

Then

$W = \int_0^{20}(490 - 24.5x)\,dx = \Big[490x - 12.25x^2\Big]_0^{20} = 4900$ J.

6. When the bucket is x m off the ground, the water weighs

$F(x) = 490\left(\frac{20 - 4x/5}{20}\right) = 490\left(1 - \frac{x}{25}\right) = 490 - 19.6x$ N.

Then

$W = \int_0^{20}(490 - 19.6x)\,dx = \Big[490x - 9.8x^2\Big]_0^{20} = 5880$ J.

7. When the bag is x ft off the ground, the sand weighs

$F(x) = 144\left(\frac{18 - x/2}{18}\right) = 144\left(1 - \frac{x}{36}\right) = 144 - 4x$ lb.

Then

$W = \int_0^{18}(144 - 4x)\,dx = \Big[144x - 2x^2\Big]_0^{18} = 1944$ ft-lb

8. **(a)** $F = ks$, so $800 = k(14 - 10)$ and $k = 200$ lb/in.

(b) $F(x) = 200x$, and $\int_0^2 200x\,dx = \Big[100x^2\Big]_0^2 = 400$ in.-lb.

(c) $F = 200s$, so $s = \frac{1600}{200} = 8$ in.

9. **(a)** $F = ks$, so $21{,}714 = k(8 - 5)$ and $k = 7238$ lb/in.

(b) $F(x) = 7238x$. $W = \int_0^{1/2} 7238x\,dx = \Big[3619x^2\Big]_0^{1/2}$

$= 904.75 \approx 905$ in.-lb, and $W = \int_{1/2}^1 7238x\,dx$

$= \Big[3619x^2\Big]_{1/2}^1 = 2714.25 \approx 2714$ in.-lb.

10. **(a)** $F = ks$, so $150 = k\left(\frac{1}{16}\right)$ and $k = 2400$ lb/in. Then for $s = \frac{1}{8}$, $F = 2400\left(\frac{1}{8}\right) = 300$ lb.

(b) $\int_0^{1/8} 2400x\,dx = \Big[1200x^2\Big]_0^{1/8} = 18.75$ in.-lb

11. When the end of the rope is x m from its starting point, the $(50 - x)$ m of rope still to go weigh

$F(x) = (0.624)(50 - x)$ N. The total work is

$\int_0^{50}(0.624)(50 - x)\,dx = 0.624\left[50x - \frac{1}{2}x^2\right]_0^{50} = 780$ J.

12. **(a)** Work $= \int_{(p_1, V_1)}^{(p_2, V_2)} F(x)\,dx = \int_{(p_1, V_1)}^{(p_2, V_2)} pA\,dx = \int_{(p_1, V_1)}^{(p_2, V_2)} p\,dV$

(b) $p_1 V_1^{\,1.4} = (50)(243)^{1.4} = 109{,}350$, so $p = \frac{109{,}350}{V^{1.4}}$ and

Work $= \int_{(p_1, V_1)}^{(p_2, V_2)} \frac{109{,}350}{V^{1.4}}\,dV$

$= 109{,}350\Big[-2.5V^{-0.4}\Big]_{V=243}^{V=32}$

$= -37{,}968.75$ in.-lb

13. **(a)** From the equation $x^2 + y^2 = 3^2$, it follows that a thin horizontal rectangle has area $2\sqrt{9 - y^2}\,\Delta y$, where y is distance from the top, and pressure $62.4y$. The total force is approximately $\sum_{k=1}^{n}(62.4y_k)(2\sqrt{9 - y_k^{\,2}})\,\Delta y$

$= \sum_{k=1}^{n} 124.8y_k\sqrt{9 - y_k^{\,2}}\,\Delta y.$

(b) $\int_0^3 124.8y\sqrt{9 - y^2}\,dy = \Big[-41.6(9 - y^2)^{3/2}\Big]_0^3$

$= 1123.2$ lb

14. **(a)** From the equation $\frac{x^2}{3^2} + \frac{y^2}{8^2} = 1$, it follows that a thin horizontal rectangle has area $6\sqrt{1 - \frac{y^2}{64}}\,\Delta y$, where y is distance from the top, and pressure $62.4y$. The total force is approximately

$\sum_{k=1}^{n}(62.4y_k)\left(6\sqrt{1 - \frac{y_k^{\,2}}{64}}\right)\Delta y = \sum_{k=1}^{n} 374.4y_k\sqrt{1 - \frac{y_k^{\,2}}{64}}\,\Delta y.$

(b) $\int_0^8 374.4y\sqrt{1 - \frac{y^2}{64}}\,dy = \left[-7987.2\left(1 - \frac{y^2}{64}\right)^{3/2}\right]_0^8$

$= 7987.2$ lb

15. (a) From the equation $x = \frac{3}{8}y$, it follows that a thin horizontal rectangle has area $\frac{3}{4}y\Delta y$, where y is the distance from the top of the triangle, the pressure is $62.4(y - 3)$. The total force is approximately

$$\sum_{k=1}^{n} 62.4(y_k - 3)\left(\frac{3}{4}y_k\right)\Delta y = \sum_{k=1}^{n} 46.8(y_k{}^2 - 3y_k)\Delta y.$$

(b) $\int_3^8 46.8(y_2{}^2 - 3y)\,dy = \Big[15.6y^3 - 70.2y^2\Big]_3^8$
$= 3494.4 - (-210.6) - 3705$ lb

16. (a) From the equation $y = \frac{x^2}{2}$, it follows that a thin horizontal rectangle has area $2\sqrt{2y}\Delta y$, where y is distance from the bottom, and pressure $62.4(4 - y)$. The total force is approximately

$$\sum_{k=1}^{n} 62.4(4 - y_k)(2\sqrt{2y_k})\Delta y$$
$$= \sum_{k=1}^{n} 124.8\sqrt{2}(4\sqrt{y_k} - y_k{}^{3/2})\Delta y.$$

(b) $\int_0^4 124.8\sqrt{2}(4\sqrt{y} - y^{3/2})\,dy$
$= 124.8\sqrt{2}\left[\frac{8}{3}y^{3/2} - \frac{2}{5}y^{5/2}\right]_0^4$
$= 1064.96\sqrt{2} \approx 1506.1$ lb

17. (a) Work to raise a thin slice $= 62.4(10 \times 12)(\Delta y)y$.
Total work $= \int_0^{20} 62.4(120)y\,dy = 62.4\Big[60y^2\Big]_0^{20}$
$= 1{,}497{,}600$ ft-lb

(b) (1,497,600 ft-lb) $\div$ (250 ft-lb/sec) = 5990.4 sec $\approx$ 100 min

(c) Work to empty half the tank $= \int_0^{10} 62.4(120)y\,dy$
$= 62.4\Big[60y^2\Big]_0^{10} = 374{,}400$ ft-lb, and
$374{,}400 \div 250 = 1497.6$ sec ≈ 25 min

(d) The weight per ft^3 of water is a simple multiplicative factor in the answers. So divide by 62.4 and multiply by the appropriate weight-density

For 62.26:

$1{,}497{,}600\left(\frac{62.26}{62.4}\right) = 1{,}494{,}240$ ft-lb and
$5990.4\left(\frac{62.26}{62.4}\right) = 5976.96$ sec ≈ 100 min.

For 62.5:

$1{,}497{,}600\left(\frac{62.5}{62.4}\right) = 1{,}500{,}000$ ft-lb and
$5990.4\left(\frac{62.5}{62.4}\right) = 6000$ sec $= 100$ min.

18. The work needed to raise a thin disk is $\pi(10)^2(51.2)y\Delta y$, where y is height up from the bottom. The total work is

$\int_0^{30} 100\pi(51.2)y\,dy = 5120\pi\left[\frac{1}{2}y^2\right]_0^{30} \approx 7{,}238{,}229$ ft-lb

19. Work to pump through the valve is $\pi(2)^2(62.4)(y + 15)\Delta y$ for a thin disk and

$\int_0^6 4\pi(62.4)(y + 15)\,dy = 249.6\pi\left[\frac{1}{2}y^2 + 15y\right]_0^6$
$\approx 84{,}687.3$ ft-lb

for the whole tank. Work to pump over the rim is $\pi(2)^2(62.4)(6 + 15)\Delta y$ for a thin disk and $\int_0^6 4\pi(62.4)(21)\,dy = 4\pi(62.4)(21)(6) \approx 98{,}801.8$ ft-lb for the whole tank. Through a hose attached to a valve in the bottom is faster, because it takes more time for a pump with a given power output to do more work.

20. The work is the same as if the straw were initially an inch long and just touched the surface, and lengthened as the liquid level dropped. For a thin disk, the volume is $\pi\left(\frac{y + 17.5}{14}\right)^2\Delta y$ and the work to raise it is $\pi\left(\frac{y + 17.5}{14}\right)^2\left(\frac{4}{9}\right)(8 - y)\Delta y$. The total work is $\int_0^7 \pi\left(\frac{y + 17.5}{14}\right)^2\left(\frac{4}{9}\right)(8 - y)\,dy$, which using NINT evaluates to ≈ 91.3244 in.-oz.

21. The work is that already calculated (to pump the oil to the rim) plus the work needed to raise the entire amount 3 ft higher. The latter comes to $\left(\frac{1}{3}\pi r^2h\right)(57)(3) = 57\pi(4)^2(8) = 22{,}921.06$ ft-lb, and the total is $22{,}921.06 + 30{,}561.41 \approx 53{,}482.5$ ft-lb.

22. The weight density is a simple multiplicative factor: Divide by 57 and multiply by 64.5.

$30{,}561.41\left(\frac{64.5}{57}\right) \approx 34{,}582.65$ ft-lb.

23. The work to raise a thin disk is

$\pi r^2(56)h = \pi(\sqrt{10^2 - y^2})^2(56)(10 + 2 - y)\Delta y$

$= 56\pi(12 - y)(100 - y^2)\Delta y$. The total work is

$\int_0^{10} 56\pi(12 - y)(100 - y^2)\, dy$, which evaluates using NINT

to $\approx$967,611 ft-lb. This will come to

(967,611)($0.005) $\approx$ $4838, so yes, there's enough money

to hire the firm.

24. Pipe radius $= \frac{1}{6}$ ft;

Work to fill pipe $= \int_0^{360} \pi\left(\frac{1}{6}\right)^2(62.4)y\, dy = 112{,}320\pi$ ft-lb.

Work to fill tank $= \int_{360}^{385} \pi(10)^2(62.4)y\, dy$

$= 58{,}110{,}000\pi$ ft-lb.

Total work $= 58{,}222{,}320\pi$ ft-lb, which will take

$58{,}222{,}320\pi \div 1650 \approx 110{,}855$ sec $\approx$ 31 hr.

25. **(a)** The pressure at depth y is $62.4y$, and the area of a thin horizontal strip is $2\Delta y$. The depth of water is $\frac{11}{6}$ ft, so the total force on an end is

$\int_0^{11/6} (62.4y)(2\, dy) \approx 209.73$ lb.

(b) On the sides, which are twice as long as the ends, the initial total force is doubled to $\approx$419.47 lb. When the tank is upended, the depth is doubled to $\frac{11}{3}$ ft, and the force on a side becomes $\int_0^{11/3} (62.4y)(2\, dy) \approx 838.93$ lb, which means that the fluid force doubles.

26. 3.75 in. $= \frac{5}{16}$ ft, and 7.75 in. $= \frac{31}{48}$ ft.

Force on a side $= \int p\, dA = \int_0^{31/48} (64.5y)\left(\frac{5}{16}\, dy\right) \approx 4.2$ lb.

27. **(a)** 0.5 (50%), since half of a normal distribution lies below the mean.

(b) Use NINT to find $\int_{63}^{65} f(x)\, dx$, where

$f(x) = \frac{1}{3.2\sqrt{2\pi}} e^{-(x-63.4)^2/(2 \cdot 3.2^2)}$. The result is $\approx$0.24 (24%).

(c) 6 ft = 72 in. Pick 82 in. as a conveniently high upper limit and with NINT, find $\int_{72}^{82} f(x)\, dx$. The result is $\approx$0.0036 (0.36%).

(d) 0 if we assume a continuous distribution. Between 59.5 in. and 60.5 in., the proportion is

$\int_{59.5}^{60.5} f(x)\, dx \approx 0.071$ (7.1%)

28. Use $f(x) = \frac{1}{100\sqrt{2\pi}} e^{-(x-498)^2/(2 \cdot 100^2)}$

(a) $\int_{400}^{500} f(x)\, dx \approx 0.34$ (34%)

(b) Take 1000 as a conveniently high upper limit:

$\int_{700}^{1000} f(x)\, dx \approx 0.217$, which means about

$0.217(300) \approx 6.5$ people

29. Integration is a good approximation to the area (which represents the probability), since the area is a kind of Riemann sum.

30. The proportion of lightbulbs that last between 100 and 800 hours.

31. $\int_{6{,}370{,}000}^{35{,}780{,}000} \frac{1000MG}{r^2}\, dr = 1000\, MG\left[-\frac{1}{r}\right]_{6{,}370{,}000}^{35{,}780{,}000}$, which for $M = 5.975 \times 10^{24}$, $G = 6.6726 \times 10^{-11}$ evaluates to

$\approx 5.1446 \times 10^{10}$ J.

32. **(a)** The distance goes from 2 m to 1 m. The work by an external force equals the work done by repulsion in moving the electrons from a 1-m distance to a 2-m distance:

$$\text{Work} = \int_1^2 \frac{23 \times 10^{-29}}{r^2}\, dr$$
$$= 23 \times 10^{-29}\left[-\frac{1}{r}\right]_1^2$$
$$= 1.15 \times 10^{-28} \text{ J}$$

(b) Again, find the work done by the fixed electrons in pushing the third one away. The total work is the sum of the work by each fixed electron. The changes in distance are 4 m to 6 m and 2 m to 4 m, respectively.

$$\text{Work} = \int_4^6 \frac{23 \times 10^{-29}}{r^2}\, dr + \int_2^4 \frac{23 \times 10^{-29}}{r^2}\, dr$$
$$= 23 \times 10^{-29}\left(\left[-\frac{1}{r}\right]_4^6 + \left[-\frac{1}{r}\right]_2^4\right)$$
$$\approx 7.6667 \times 10^{-29} \text{ J.}$$

33. $F = m\left(\frac{dv}{dt}\right) = mv\left(\frac{dv}{dx}\right)$, so

$$W = \int_{x_1}^{x_2} F(x)\,dx$$
$$= \int_{x_1}^{x_2} mv\left(\frac{dv}{dx}\right)dx$$
$$= \int_{v_1}^{v_2} mv\,dv$$
$$= \frac{1}{2}mv_2^{\,2} - \frac{1}{2}mv_1^{\,2}$$

34. Work = Change in kinetic energy $= \frac{1}{2}mv^2$.

$m = \frac{2\text{ oz}}{32\text{ ft/sec}^2} = \frac{1/8\text{ lb}}{32\text{ ft/sec}^2} = \frac{1}{256}$ slug, so

Work $= \frac{1}{2}\left(\frac{1}{256}\right)(160)^2 = 50$ ft-lb.

35. $0.3125\text{ lb} = \frac{0.3125\text{ lb}}{32\text{ ft/sec}^2} = 0.009765625$ slug, and

$90\text{ mph} = 90\left(\frac{5280\text{ ft}}{1\text{ mi}}\right)\left(\frac{1\text{ hr}}{3600\text{ sec}}\right) = 132$ ft/sec, so

Work = change in kinetic energy $= \frac{1}{2}(0.009765625)(132)^2$

≈ 85.1 ft-lb.

36. $1.6\text{ oz} = 1.6\text{ oz}\left(\frac{1\text{ lb}}{16\text{ oz}}\right)/(32\text{ ft/sec}^2) = 0.003125$ slug, so

Work $= \frac{1}{2}(0.003125)(280)^2 = 122.5$ ft-lb.

37. $2\text{ oz} = 2\text{ oz}\left(\frac{1\text{ lb}}{16\text{ oz}}\right)/(32\text{ ft/sec}^2) = \frac{1}{256}$ slug, and

$124\text{ mph} = 124\text{ mph}\left(\frac{5280\text{ ft}}{1\text{ mi}}\right)\left(\frac{1\text{ hr}}{3600\text{ sec}}\right) = 181.867$ ft/sec,

so Work $= \frac{1}{2}\left(\frac{1}{256}\right)(181.867)^2 \approx 64.6$ ft-lb.

38. $14.5\text{ oz} = 14.5\text{ oz}\left(\frac{1\text{ lb}}{16\text{ oz}}\right)/(32\text{ ft/sec}^2) \approx 0.02832$ slug, so

Work $= \frac{1}{2}(0.02832)(88)^2 \approx 109.7$ ft-lb.

39. $6.5\text{ oz} = 6.5\text{ oz}\left(\frac{1\text{ lb}}{16\text{ oz}}\right)/(32\text{ ft/sec}^2) \approx 0.01270$ slug, so

Work $= \frac{1}{2}(0.01270)(132)^2 \approx 110.6$ ft-lb.

40. $2\text{ oz} = \frac{1}{8}\text{ lb} = \frac{1}{256}$ slug. Compression energy of spring $= \frac{1}{2}ks^2 = \frac{1}{2}(18)\left(\frac{1}{4}\right)^2 = 0.5625$ ft-lb, and final height is given by $mgh = 0.5625$ ft-lb, so $h = \frac{0.5625}{(1/256)(32)} = 4.5$ ft.

■ Chapter 7 Review Exercises

(pp. 413–415)

1. $\int_0^5 v(t)\,dt = \int_0^5 (t^2 - 0.2t^3)\,dt$

$= \left[\frac{1}{3}t^3 - 0.05t^4\right]_0^5 \approx 10.417$ ft

2. $\int_0^7 c(t)\,dt = \int_0^7 (4 + 0.001t^4)\,dt$

$= \left[4t + 0.0002t^5\right]_0^7 \approx 31.361$ gal

3. $\int_0^{100} B(x)\,dx = \int_0^{100} (21 - e^{0.03x})\,dx$

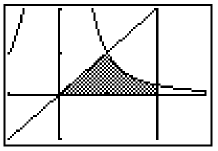
$\approx \left[21x - 33.333e^{0.03x}\right]_0^{100} \approx 1464$

4. $\int_0^2 \rho(x)\,dx = \int_0^2 (11 - 4x)\,dx = \left[11x - 2x^2\right]_0^2 = 14$ g

5. $\int_0^{24} E(t)\,dt = \int_0^{24} 300\left(2 - \cos\left(\frac{\pi t}{12}\right)\right)dt$

$= 300\left[2t - \frac{12}{\pi}\sin\left(\frac{\pi t}{12}\right)\right]_0^{24} = 14{,}400$

6.



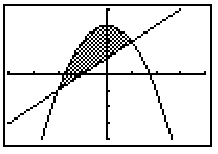
[−1, 3] by [−1, 2]

The curves intersect at $x = 1$. The area is

$\int_0^1 x\,dx + \int_1^2 \frac{1}{x^2}\,dx = \left[\frac{1}{2}x^2\right]_0^1 + \left[-\frac{1}{x}\right]_1^2$

$= \frac{1}{2} + \left(-\frac{1}{2} + 1\right) = 1.$

7.



[−4, 4] by [−4, 4]

The curves intersect at $x = -2$ and $x = 1$. The area is

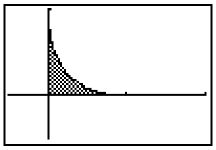
$\int_{-2}^1 [3 - x^2 - (x + 1)]\,dx = \int_{-2}^1 (-x^2 - x + 2)\,dx$

$= \left[-\frac{1}{3}x^3 - \frac{1}{2}x^2 + 2x\right]_{-2}^1$

$= \left(-\frac{1}{3} - \frac{1}{2} + 2\right) - \left(\frac{8}{3} - 2 - 4\right)$

$= \frac{9}{2}.$

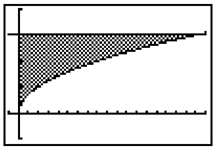
8. $\sqrt{x} + \sqrt{y} = 1$ implies $y = (1 - \sqrt{x})^2 = 1 - 2\sqrt{x} + x$.



$[-0.5, 2]$ by $[-0.5, 1]$

The area is $\int_0^1 (1 - 2\sqrt{x} + x)\, dx = \left[x - \frac{4}{3}x^{3/2} + \frac{1}{2}x^2\right]_0^1 = \frac{1}{6}.$

9. $x = 2y^2$ implies $y = \sqrt{\frac{x}{2}}$.



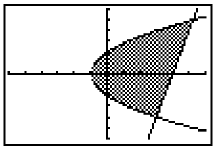
$[-1, 19]$ by $[-1, 4]$

The curves intersect at $x = 18$. The area is

$$\int_0^{18}\left(3 - \sqrt{\frac{x}{2}}\right) dx = \left[3x - \frac{4}{3}\left(\frac{x}{2}\right)^{3/2}\right]_0^{18} = 18,$$

or $\int_0^3 2y^2\, dy = \left[\frac{2}{3}y^3\right]_0^3 = 18.$

10. $4x = y^2 - 4$ implies $x = \frac{1}{4}y^2 - 1$, and $4x = y + 16$ implies $x = \frac{1}{4}y + 4$.



$[-6, 6]$ by $[-6, 6]$

The curves intersect at $(3, -4)$ and $(5.25, 5)$. The area is

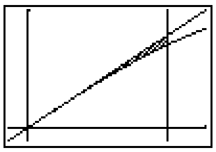
$$\int_{-4}^{5}\left[\left(\frac{1}{4}y + 4\right) - \left(\frac{1}{4}y^2 - 1\right)\right] dy$$

$$= \int_{-4}^{5}\left(-\frac{1}{4}y^2 + \frac{1}{4}y + 5\right) dy$$

$$= \left[-\frac{1}{12}y^3 + \frac{1}{8}y^2 + 5y\right]_{-4}^{5}$$

$$= \frac{425}{24} - \left(-\frac{38}{3}\right) = \frac{243}{8} = 30.375.$$

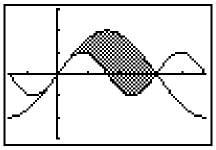
11.



$[-0.1, 1]$ by $[-0.1, 1]$

The area is $\int_0^{\pi/4}(x - \sin x)\, dx = \left[\frac{1}{2}x^2 + \cos x\right]_0^{\pi/4} = \frac{\pi^2}{32} + \frac{\sqrt{2}}{2} - 1 \approx 0.0155.$

12.

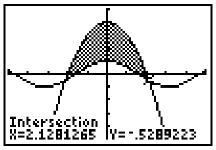


$\left[-\frac{\pi}{2}, \frac{3\pi}{2}\right]$ by $[-3, 3]$

The area is

$$\int_0^{\pi}(2\sin x - \sin 2x)\, dx = \left[-2\cos x + \frac{1}{2}\cos 2x\right]_0^{\pi} = 4.$$

13.



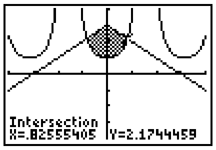
$[-5, 5]$ by $[-5, 5]$

The curves intersect at $x \approx \pm 2.1281$. The area is

$$\int_{-2.1281}^{2.1281}(4 - x^2 - \cos x)\, dx,$$

which using NINT evaluates to ≈ 8.9023.

14.



$[-4, 4]$ by $[-4, 4]$

The curves intersect at $x \approx \pm 0.8256$. The area is

$$\int_{-0.8256}^{0.8256}(3 - |x| - \sec^2 x)\, dx,$$

which using NINT evaluates to ≈ 2.1043.

15. Solve $1 + \cos x = 2 - \cos x$ for the x-values at the two ends of the region: $x = 2\pi \pm \frac{\pi}{3}$, i.e., $\frac{5\pi}{3}$ or $\frac{7\pi}{3}$. Use the symmetry of the area:

$$2\int_{2\pi}^{7\pi/3}[(1 + \cos x) - (2 - \cos x)]\, dx$$

$$= 2\int_{2\pi}^{7\pi/3}(2\cos x - 1)\, dx$$

$$= 2\Big[2\sin x - x\Big]_{2\pi}^{7\pi/3}$$

$$= 2\sqrt{3} - \frac{2}{3}\pi \approx 1.370.$$

16. $\int_{\pi/3}^{5\pi/3}[(2 - \cos x) - (1 + \cos x)]\, dx$

$$= \int_{\pi/3}^{5\pi/3}(1 - 2\cos x)\, dx$$

$$= \Big[x - 2\sin x\Big]_{\pi/3}^{5\pi/3}$$

$$= 2\sqrt{3} + \frac{4}{3}\pi \approx 7.653$$

17. Solve $x^3 - x = \frac{x}{x^2+1}$ to find the intersection points at $x = 0$ and $x = \pm 2^{1/4}$. Then use the area's symmetry: the area is

$$2\int_0^{2^{1/4}} \left[\frac{x}{x^2+1} - (x^3 - x)\right] dx$$
$$= 2\left[\frac{1}{2}\ln(x^2+1) - \frac{1}{4}x^4 + \frac{1}{2}x^2\right]_0^{2^{1/4}}$$
$$= \ln(\sqrt{2}+1) + \sqrt{2} - 1 \approx 1.2956.$$

18. Use the intersect function on a graphing calculator to determine that the curves intersect at $x \approx \pm 1.8933$. The area is

$$\int_{-1.8933}^{1.8933} \left(3^{1-x^2} - \frac{x^2-3}{10}\right) dx,$$

which using NINT evaluates to ≈ 5.7312.

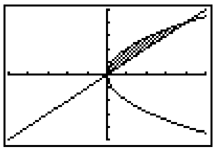
19. Use the x- and y-axis symmetries of the area:

$$4\int_0^{\pi} x \sin x\, dx = 4\Big[\sin x - x\cos x\Big]_0^{\pi} = 4\pi.$$

20. A cross section has radius $r = 3x^4$ and area $A(x) = \pi r^2 = 9\pi x^8$.

$$V = \int_{-1}^{1} 9\pi x^8\, dx = \pi\Big[x^9\Big]_{-1}^{1} = 2\pi.$$

21.



[−5, 5] by [−5, 5]

The graphs intersect at (0, 0) and (4, 4).

(a) Use cylindrical shells. A shell has radius y and height $y - \frac{y^2}{4}$. The total volume is

$$\int_0^4 2\pi(y)\left(y - \frac{y^2}{4}\right) dy = 2\pi\int_0^4 \left(y^2 - \frac{y^3}{4}\right) dy$$
$$= 2\pi\left[\frac{1}{3}y^3 - \frac{1}{16}y^4\right]_0^4$$
$$= \frac{32\pi}{3}.$$

(b) Use cylindrical shells. A shell has radius x and height $2\sqrt{x} - x$. The total volume is

$$\int_0^4 2\pi(x)(2\sqrt{x} - x)\, dx = 2\pi\int_0^4 (2x^{3/2} - x^2)\, dx$$
$$= 2\pi\left[\frac{4}{5}x^{5/2} - \frac{1}{3}x^3\right]_0^4$$
$$= \frac{128\pi}{15}.$$

(c) Use cylindrical shells. A shell has radius $4 - x$ and height $2\sqrt{x} - x$. The total volume is

$$\int_0^4 2\pi(4-x)(2\sqrt{x} - x)\, dx$$
$$= 2\pi\int_0^4 (8\sqrt{x} - 4x - 2x^{3/2} + x^2)\, dx$$
$$= 2\pi\left[\frac{16}{3}x^{3/2} - 2x^2 - \frac{4}{5}x^{5/2} + \frac{1}{3}x^3\right]_0^4 = \frac{64\pi}{5}.$$

(d) Use cylindrical shells. A shell has radius $4 - y$ and height $y - \frac{y^2}{4}$. The total volume is

$$\int_0^4 2\pi(4-y)\left(y - \frac{y^2}{4}\right) dy$$
$$= 2\pi\int_0^4 \left(4y - 2y^2 + \frac{y^3}{4}\right) dy$$
$$= 2\pi\left[2y^2 - \frac{2}{3}y^3 + \frac{1}{16}y^4\right]_0^4 = \frac{32\pi}{3}.$$

22. (a) Use disks. The volume is

$$\pi\int_0^2 (\sqrt{2y})^2\, dy = \pi\int_0^2 2y\, dy = \pi y^2\Big]_0^2 = 4\pi.$$

(b) $\pi\int_0^k 2y\, dy = \pi y^2\Big]_0^k = \pi k^2$

(c) Since $V = \pi k^2$, $\frac{dV}{dt} = 2\pi k\frac{dk}{dt}$.

When $k = 1$, $\frac{dk}{dt} = \frac{1}{2\pi k}\frac{dV}{dt} = \left(\frac{1}{2\pi}\right)(2) = \frac{1}{\pi}$,

so the depth is increasing at the rate of $\frac{1}{\pi}$ unit per second.

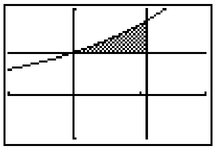
23. The football is a solid of revolution about the x-axis. A cross section has radius $\sqrt{12\left(1 - \frac{4x^2}{121}\right)}$ and area $\pi r^2 = 12\pi\left(1 - \frac{4x^2}{121}\right)$. The volume is, given the symmetry,

$$2\int_0^{11/2} 12\pi\left(1 - \frac{4x^2}{121}\right) dx = 24\pi\int_0^{11/2}\left(1 - \frac{4x^2}{121}\right) dx$$
$$= 24\pi\left[x - \left(\frac{2}{11}\right)^2\left(\frac{1}{3}\right)x^3\right]_0^{11/2}$$
$$= 24\pi\left[\frac{11}{2} - \frac{11}{6}\right]$$
$$= 88\pi \approx 276 \text{ in}^3.$$

24. The width of a cross section is $2\sin x$, and the area is $\frac{1}{2}\pi r^2 = \frac{1}{2}\pi\sin^2 x$. The volume is

$$\int_0^{\pi} \frac{1}{2}\pi\sin^2 x\, dx = \frac{\pi}{2}\left[\frac{1}{2}x - \frac{1}{4}\sin 2x\right]_0^{\pi} = \frac{\pi^2}{4}.$$

25.



$[-1, 2]$ by $[-1, 2]$

Use washer cross sections. A washer has inner radius $r = 1$, outer radius $R = e^{x/2}$, and area $\pi(R^2 - r^2) = \pi(e^x - 1)$.

The volume is

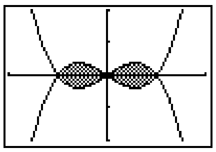
$$\int_0^{\ln 3} \pi(e^x - 1)\,dx = \pi\Big[e^x - x\Big]_0^{\ln 3}$$
$$= \pi(3 - \ln 3 - 1)$$
$$= \pi(2 - \ln 3).$$

26. Use cylindrical shells. Taking the hole to be vertical, a shell has radius x and height $2\sqrt{2^2 - x^2}$. The volume of the piece cut out is

$$\int_0^{\sqrt{3}} 2\pi(x)(2\sqrt{2^2 - x^2})\,dx = 2\pi\int_0^{\sqrt{3}} 2x\sqrt{4 - x^2}\,dx$$
$$= 2\pi\left[-\frac{2}{3}(4 - x^2)^{3/2}\right]_0^{\sqrt{3}}$$
$$= -\frac{4}{3}\pi(1 - 8)$$
$$= \frac{28\pi}{3} \approx 29.3215 \text{ ft}^3.$$

27. The curve crosses the x-axis at $x = \pm 3$. $y' = -2x$, so the length is $\int_{-3}^{3}\sqrt{1 + (-2x)^2}\,dx = \int_{-3}^{3}\sqrt{1 + 4x^2}\,dx$, which using NINT evaluates to ≈ 19.4942.

28.

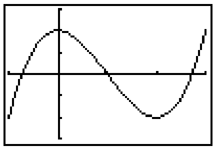


$[-2, 2]$ by $[-2, 2]$

The curves intersect at $x = 0$ and $x = \pm 1$. Use the graphs' x- and y-axis symmetry:

$\frac{d}{dx}(x^3 - x) = 3x^2 - 1$, and the total perimeter is $4\int_0^1 \sqrt{1 + (3x^2 - 1)^2}\,dx$, which using NINT evaluates to ≈ 5.2454.

29.



$[-1, 3]$ by $[-3, 3]$

$y' = 3x^2 - 6x$ equals zero when $x = 0$ or 2. The maximum is at $x = 0$, the minimum at $x = 2$. The distance between them along the curve is $\int_0^2 \sqrt{1 + (3x^2 - 6x)^2}\,dx$, which using NINT evaluates to ≈ 4.5920. The time taken is about $\frac{4.5920}{2} = 2.296$ sec.

30. If (b) were true, then the curve $y = k \sin x$ would have to get vanishingly short as k approached zero. Since in fact the curve's length approaches 2π instead, (b) is false and (a) is true.

31. $F'(x) = \sqrt{x^4 - 1}$, so

$$\int_2^5 \sqrt{1 + (F'(x))^2}\,dx = \int_2^5 \sqrt{x^4}\,dx$$
$$= \int_2^5 x^2\,dx$$
$$= \left[\frac{1}{3}x^3\right]_2^5 = 39.$$

32. **(a)** (100 N)(40 m) = 4000 J

(b) When the end has traveled a distance y, the weight of the remaining portion is $(40 - y)(0.8) = 32 - 0.8y$.

The total work to lift the rope is

$$\int_0^{40} (32 - 0.8y)\,dy = \Big[32y - 0.4y^2\Big]_0^{40} = 640 \text{ J}.$$

(c) $4000 + 640 = 4640$ J

33. The weight of the water at elevation x (starting from $x = 0$) is $(800)(8)\left(\frac{4750 - x/2}{4750}\right) = \frac{128}{95}\left(4750 - \frac{1}{2}x\right)$. The total work is $\int_0^{4750} \frac{128}{95}\left(4750 - \frac{1}{2}x\right)dx = \frac{128}{95}\left[4750x - \frac{1}{4}x^2\right]_0^{4750}$

$$= 22{,}800{,}000 \text{ ft-lb}.$$

34. $F = ks$, so $k = \frac{F}{s} = \frac{80}{0.3} = \frac{800}{3}$ N/m. Then

$$\text{Work} = \int_0^{0.3} \frac{800}{3}x\,dx = \left[\frac{800}{6}x^2\right]_0^{0.3} = 12 \text{ J}.$$

To stretch the additional meter,

$$\text{Work} = \int_{0.3}^{1.3} \frac{800}{3}x\,dx = \left[\frac{800}{6}x^2\right]_{0.3}^{1.3} \approx 213.3 \text{ J}.$$

35. The work is positive going uphill, since the force pushes in the direction of travel. The work is negative going downhill.

36. The radius of a horizontal cross section is $\sqrt{8^2 - y^2}$, where y is distance below the rim. The area is $\pi(64 - y^2)$, the weight is $0.04\pi(64 - y^2)\,\Delta y$, and the work to lift it over the rim is $0.04\pi(64 - y^2)(y)\,\Delta y$. The total work is

$$\int_2^8 0.04\pi y(64 - y^2)\,dy = 0.04\pi\int_2^8 (64y - y^3)\,dy = 0.04\pi\left[32y^2 - \frac{1}{4}y^4\right]_2^8 = 36\pi \approx 113.097 \text{ in.-lb.}$$

37. The width of a thin horizontal strip is $2(2y) = 4y$, and the force against it is $80(2 - y)4y\,\Delta y$. The total force is

$$\int_0^2 320y(2 - y)\,dy = 320\int_0^2 (-y^2 + 2y)\,dy = 320\left[-\frac{1}{3}y^3 + y^2\right]_0^2 = \frac{1280}{3} \approx 426.67 \text{ lb.}$$

38. 5.75 in. $= \frac{23}{48}$ ft, 3.5 in. $= \frac{7}{24}$ ft, and 10 in. $= \frac{5}{6}$ ft.

For the base,

$$\text{Force} = 57\left(\frac{23}{48} \times \frac{7}{24} \times \frac{5}{6}\right) \approx 6.6385 \text{ lb.}$$

For the front and back,

$$\text{Force} = \int_0^{5/6} 57\left(\frac{7}{24}\right)y\,dy = \frac{399}{24}\left[\frac{1}{2}y^2\right]_0^{5/6} \approx 5.7726 \text{ lb.}$$

For the sides,

$$\text{Force} = \int_0^{5/6} 57\left(\frac{23}{48}\right)y\,dy = \frac{1311}{48}\left[\frac{1}{2}y^2\right]_0^{5/6} \approx 9.4835 \text{ lb.}$$

39. A square's height is $y = (\sqrt{6} - \sqrt{x})^2$, and its area is $y^2 = (\sqrt{6} - \sqrt{x})^4$. The volume is $\int_0^6 (\sqrt{6} - \sqrt{x})^4\,dx$, which using NINT evaluates to exactly 14.4.

40. Choose 50 cm as a conveniently large upper limit.

$\int_{20}^{50} \frac{1}{3.4\sqrt{2\pi}}\, e^{-(x-17.2)^2/(2\cdot 3.4^2)}\,dx$, evaluates, using NINT to ≈ 0.2051 (20.5%).

41. Answers will vary. Find μ, then use the fact that 68% of the class is within σ of μ to find σ, and then choose a conveniently large number b and calculate

$$\int_{10}^{b} \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/(2\sigma^2)}\,dx.$$

42. Use $f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$.

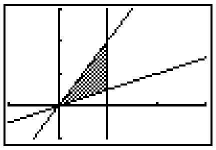
(a) $\int_{-1}^{1} f(x)\,dx$ evaluates, using NINT, to ≈ 0.6827 (68.27%).

(b) $\int_{-2}^{2} f(x)\,dx \approx 0.9545$ (95.45%)

(c) $\int_{-3}^{3} f(x)\,dx \approx 0.9973$ (99.73%)

43. Because $f(x) \ge 0$ and $\int_{-\infty}^{\infty} f(x)\,dx = 1$

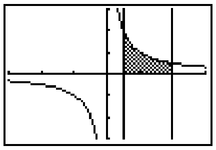
44.



$[-1, 3]$ by $[-1, 3]$

A shell has radius x and height $2x - \frac{x}{2} = \frac{3}{2}x$. The total volume is $\int_0^1 2\pi(x)\left(\frac{3}{2}x\right)dx = \pi\left[x^3\right]_0^1 = \pi$.

45.

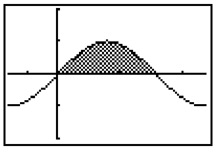


$[-3, 3]$ by $[-3, 3]$

A shell has radius x and height $\frac{1}{x}$. The total volume is

$$\int_{1/2}^{2} 2\pi(x)\left(\frac{1}{x}\right)dx = \int_{1/2}^{2} 2\pi\,dx = \left[2\pi x\right]_{1/2}^{2} = 3\pi.$$

46.

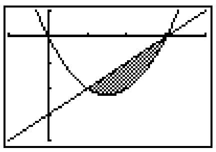


$\left[-\frac{\pi}{2}, \frac{3\pi}{2}\right]$ by $[-2, 2]$

A shell has radius x and height $\sin x$. The total volume is

$$\int_0^{\pi} 2\pi(x)(\sin x)\,dx = 2\pi\left[\sin x - x\cos x\right]_0^{\pi} = 2\pi^2.$$

47.



$[-1, 4]$ by $[-4, 1]$

The curves intersect at $x = 1$ and $x = 3$. A shell has radius x and height $x - 3 - (x^2 - 3x) = -x^2 + 4x - 3$. The total volume is

$$\int_1^3 2\pi(x)(-x^2 + 4x - 3)\,dx = 2\pi\int_1^3 (-x^3 + 4x^2 - 3x)\,dx = 2\pi\left[-\frac{1}{4}x^4 + \frac{4}{3}x^3 - \frac{3}{2}x^2\right]_1^3 = \frac{16\pi}{3}.$$

48. Use the intersect function on a graphing calculator to determine that the curves intersect at $x = \pm 1.8933$. A shell has radius x and height $3^{1-x^2} - \frac{x^2 - 3}{10}$. The volume, which is calculated using the *right half* of the area, is $\int_0^{1.8933} 2\pi(x)\left(3^{1-x^2} - \frac{x^2 - 3}{10}\right) dx$, which using NINT evaluates to ≈ 9.7717.

49. (a) $y = -\frac{5}{4}(x + 2)(x - 2) = 5 - \frac{5}{4}x^2$

(b) Revolve about the line $x = 4$, using cylindrical shells. A shell has radius $4 - x$ and height $5 - \frac{5}{4}x^2$. The total volume is

$$\int_{-2}^{2} 2\pi(4 - x)\left(5 - \frac{5}{4}x^2\right) dx$$
$$= 10\pi\int_{-2}^{2}\left(\frac{1}{4}x^3 - x^2 - x + 4\right) dx$$
$$= 10\pi\left[\frac{1}{16}x^4 - \frac{1}{3}x^3 - \frac{1}{2}x^2 + 4x\right]_{-2}^{2}$$
$$= \frac{320}{3}\pi \approx 335.1032 \text{ in}^3.$$

50. Since $\frac{dL}{dx} = \frac{1}{x} + f'(x)$ must equal $\sqrt{1 + (f'(x))^2}$, $1 + (f'(x))^2 = \frac{1}{x^2} + \frac{2}{x}f'(x) + (f'(x))^2$, and $f'(x) = \frac{1}{2}x - \frac{1}{2x}$. Then $f(x) = \frac{1}{4}x^2 - \frac{1}{2}\ln x + C$, and the requirement to pass through $(1, 1)$ means that $C = \frac{3}{4}$. The function is $f(x) = \frac{1}{4}x^2 - \frac{1}{2}\ln x + \frac{3}{4} = \frac{x^2 - 2\ln x + 3}{4}$.

51. $y' = \sec^2 x$, so the area is $\int_0^{\pi/4} 2\pi(\tan x)\sqrt{1 + (\sec^2 x)^2}\, dx$, which using NINT evaluates to ≈ 3.84.

52. $x = \frac{1}{y}$ and $x' = -\frac{1}{y^2}$, so the area is $\int_1^2 2\pi\left(\frac{1}{y}\right)\sqrt{1 + \left(-\frac{1}{y^2}\right)^2}\, dy$, which using NINT evaluates to ≈ 5.02.

Chapter 8
L'Hôpital's Rule, Improper Integrals, and Partial Fractions

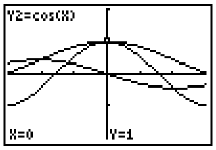
■ Section 8.1 L'Hôpital's Rule (pp. 417–425)

Exploration 1 Exploring L'Hôpital's Rule Graphically

1. $\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \frac{\cos x}{1} = 1$

2. The two graphs suggest that $\lim_{x\to 0} \frac{y_1}{y_2} = \lim_{x\to 0} \frac{y_1'}{y_2'}$.

3. $y_5 = \frac{x\cos x - \sin x}{x^2}$. The graphs of y_3 and y_5 clearly show that l'Hôpital's Rule does not say that $\lim_{x\to 0} \frac{y_1}{y_2}$ is equal to $\lim_{x\to 0} \left(\frac{y_1}{y_2}\right)'$.



$[-3, 3]$ by $[-2, 2]$

Quick Review 8.1

1.

x	$\left(1 + \frac{0.1}{x}\right)^x$
1	1.1000
10	1.1046
100	1.1051
1000	1.1052
10,000	1.1052
1,000,000	1.1052

As $x \to \infty$, $\left(1 + \frac{0.1}{x}\right)^x$ approaches 1.1052.

2.

x	$x^{1/(\ln x)}$
0.1	2.7183
0.01	2.7183
0.001	2.7183
0.0001	2.7183
0.00001	2.7183

As $x \to 0^+$, $x^{1/(\ln x)}$ approaches 2.7183.

3.

x	$\left(1 - \frac{1}{x}\right)^x$
−1	0.5
−0.1	0.78679
−0.01	0.95490
−0.001	0.99312
−0.0001	0.99908
−0.00001	0.99988
−0.000001	0.99999

As $x \to 0^-$, $\left(1 - \frac{1}{x}\right)^x$ approaches 1.

4.

x	$\left(1 + \frac{1}{x}\right)^x$
−1.1	13.981
−1.01	105.77
−1.001	1007.9
−1.0001	10010

As $x \to -1^-$, $\left(1 + \frac{1}{x}\right)^x$ goes to ∞.