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6.1 Law of Sines

What you should learn

- Use the Law of Sines to solve oblique triangles (AAS or ASA).
- Use the Law of Sines to solve oblique triangles (SSA).
- Find the areas of oblique triangles.
- Use the Law of Sines to model and solve real-life problems.

Why you should learn it

You can use the Law of Sines to solve real-life problems involving oblique triangles. For instance, in Exercise 44 on page 438, you can use the Law of Sines to determine the length of the shadow of the Leaning Tower of Pisa.

Hideo Kurihara/Getty Images

The *HM mathSpace*® CD-ROM and *Eduspace*® for this text contain additional resources related to the concepts discussed in this chapter.

Introduction

In Chapter 4, you studied techniques for solving right triangles. In this section and the next, you will solve **oblique triangles**—triangles that have no right angles. As standard notation, the angles of a triangle are labeled A, B, and C, and their opposite sides are labeled a, b, and c, as shown in Figure 6.1.

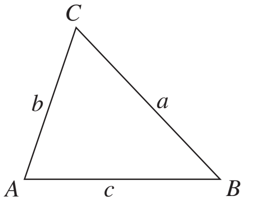


FIGURE 6.1

To solve an oblique triangle, you need to know the measure of at least one side and any two other parts of the triangle—either two sides, two angles, or one angle and one side. This breaks down into the following four cases.

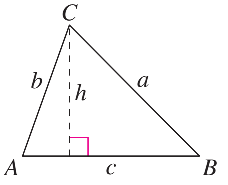
1. Two angles and any side (AAS or ASA)
2. Two sides and an angle opposite one of them (SSA)
3. Three sides (SSS)
4. Two sides and their included angle (SAS)

The first two cases can be solved using the **Law of Sines,** whereas the last two cases require the Law of Cosines (see Section 6.2).

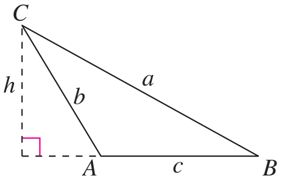
Law of Sines

If ABC is a triangle with sides a, b, and c, then

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$



A is acute.



A is obtuse.

The Law of Sines can also be written in the reciprocal form

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.$$

For a proof of the Law of Sines, see Proofs in Mathematics on page 489.

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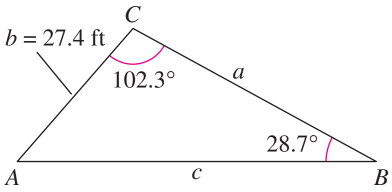

FIGURE 6.2

Example 1 Given Two Angles and One Side—AAS

For the triangle in Figure 6.2, $C = 102.3°$, $B = 28.7°$, and $b = 27.4$ feet. Find the remaining angle and sides.

Solution

The third angle of the triangle is

$$\begin{aligned} A &= 180° - B - C \\ &= 180° - 28.7° - 102.3° \\ &= 49.0°. \end{aligned}$$

By the Law of Sines, you have

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

Using $b = 27.4$ produces

$$a = \frac{b}{\sin B}(\sin A) = \frac{27.4}{\sin 28.7°}(\sin 49.0°) \approx 43.06 \text{ feet}$$

and

$$c = \frac{b}{\sin B}(\sin C) = \frac{27.4}{\sin 28.7°}(\sin 102.3°) \approx 55.75 \text{ feet.}$$

CHECKPOINT Now try Exercise 1.

> **STUDY TIP**
>
> When solving triangles, a careful sketch is useful as a quick test for the feasibility of an answer. Remember that the longest side lies opposite the largest angle, and the shortest side lies opposite the smallest angle.

Example 2 Given Two Angles and One Side—ASA

A pole tilts *toward* the sun at an 8° angle from the vertical, and it casts a 22-foot shadow. The angle of elevation from the tip of the shadow to the top of the pole is 43°. How tall is the pole?

Solution

From Figure 6.3, note that $A = 43°$ and $B = 90° + 8° = 98°$. So, the third angle is

$$\begin{aligned} C &= 180° - A - B \\ &= 180° - 43° - 98° \\ &= 39°. \end{aligned}$$

By the Law of Sines, you have

$$\frac{a}{\sin A} = \frac{c}{\sin C}.$$

Because $c = 22$ feet, the length of the pole is

$$a = \frac{c}{\sin C}(\sin A) = \frac{22}{\sin 39°}(\sin 43°) \approx 23.84 \text{ feet.}$$

CHECKPOINT Now try Exercise 35.

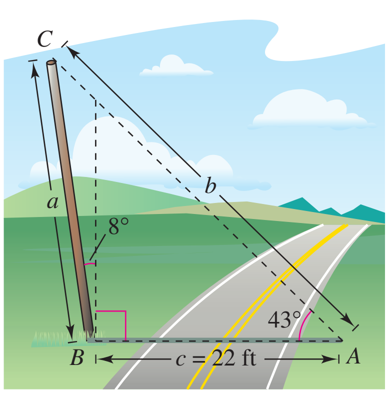

FIGURE 6.3

For practice, try reworking Example 2 for a pole that tilts *away from* the sun under the same conditions.

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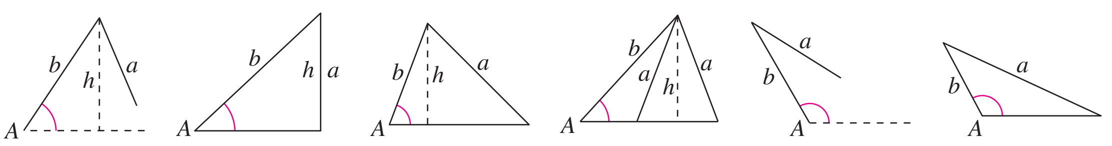
The Ambiguous Case (SSA)

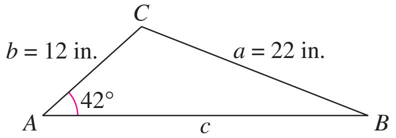
In Examples 1 and 2 you saw that two angles and one side determine a unique triangle. However, if two sides and one opposite angle are given, three possible situations can occur: (1) no such triangle exists, (2) one such triangle exists, or (3) two distinct triangles may satisfy the conditions.

The Ambiguous Case (SSA)

Consider a triangle in which you are given a, b, and A. $(h = b \sin A)$

	A is acute.	A is acute.	A is acute.	A is acute.	A is obtuse.	A is obtuse.
Sketch						
Necessary condition	$a < h$	$a = h$	$a > b$	$h < a < b$	$a \le b$	$a > b$
Triangles possible	None	One	One	Two	None	One

Example 3 Single-Solution Case—SSA



One solution: $a > b$

FIGURE 6.4

For the triangle in Figure 6.4, $a = 22$ inches, $b = 12$ inches, and $A = 42°$. Find the remaining side and angles.

Solution

By the Law of Sines, you have

$$\frac{\sin B}{b} = \frac{\sin A}{a} \qquad \text{Reciprocal form}$$

$$\sin B = b\left(\frac{\sin A}{a}\right) \qquad \text{Multiply each side by } b.$$

$$\sin B = 12\left(\frac{\sin 42°}{22}\right) \qquad \text{Substitute for } A, a, \text{ and } b.$$

$$B \approx 21.41°. \qquad B \text{ is acute.}$$

Now, you can determine that

$$C \approx 180° - 42° - 21.41° = 116.59°.$$

Then, the remaining side is

$$\frac{c}{\sin C} = \frac{a}{\sin A}$$

$$c = \frac{a}{\sin A}(\sin C) = \frac{22}{\sin 42°}(\sin 116.59°) \approx 29.40 \text{ inches.}$$

CHECKPOINT Now try Exercise 19.

Encourage your students to sketch the triangle, keeping in mind that the longest side lies opposite the largest angle of the triangle. For practice, suggest that students also find h.

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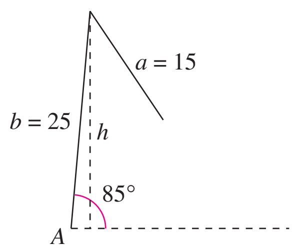

No solution: $a < h$

FIGURE 6.5

Example 4 No-Solution Case—SSA

Show that there is no triangle for which $a = 15$, $b = 25$, and $A = 85°$.

Solution

Begin by making the sketch shown in Figure 6.5. From this figure it appears that no triangle is formed. You can verify this using the Law of Sines.

$$\frac{\sin B}{b} = \frac{\sin A}{a}$$ Reciprocal form

$$\sin B = b\left(\frac{\sin A}{a}\right)$$ Multiply each side by b.

$$\sin B = 25\left(\frac{\sin 85°}{15}\right) \approx 1.660 > 1$$

This contradicts the fact that $|\sin B| \leq 1$. So, no triangle can be formed having sides $a = 15$ and $b = 25$ and an angle of $A = 85°$.

CHECKPOINT Now try Exercise 21.

Activities

Have your students determine the number of triangles possible in each of the following cases.

1. $A = 62°$, $a = 10$, $b = 12$ (0 triangles)
2. $A = 98°$, $a = 10$, $b = 3$ (1 triangle)
3. $A = 54°$, $a = 7$, $b = 10$ (0 triangles)

Discuss several examples of the two-solution case.

Additional Example

Find two triangles for which $c = 29$, $b = 46$, and $C = 31°$.

Solution

$B = 54.8°$, $A = 94.2°$, $a = 56.2$
$B = 125.2°$, $A = 23.8°$, $a = 22.7$

Example 5 Two-Solution Case—SSA

Find two triangles for which $a = 12$ meters, $b = 31$ meters, and $A = 20.5°$.

Solution

By the Law of Sines, you have

$$\frac{\sin B}{b} = \frac{\sin A}{a}$$ Reciprocal form

$$\sin B = b\left(\frac{\sin A}{a}\right) = 31\left(\frac{\sin 20.5°}{12}\right) \approx 0.9047.$$

There are two angles $B_1 \approx 64.8°$ and $B_2 \approx 180° - 64.8° = 115.2°$ between 0° and 180° whose sine is 0.9047. For $B_1 \approx 64.8°$, you obtain

$$C \approx 180° - 20.5° - 64.8° = 94.7°$$

$$c = \frac{a}{\sin A}(\sin C) = \frac{12}{\sin 20.5°}(\sin 94.7°) \approx 34.15 \text{ meters.}$$

For $B_2 \approx 115.2°$, you obtain

$$C \approx 180° - 20.5° - 115.2° = 44.3°$$

$$c = \frac{a}{\sin A}(\sin C) = \frac{12}{\sin 20.5°}(\sin 44.3°) \approx 23.93 \text{ meters.}$$

The resulting triangles are shown in Figure 6.6.

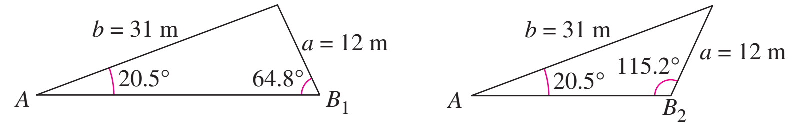

FIGURE 6.6

CHECKPOINT Now try Exercise 23.

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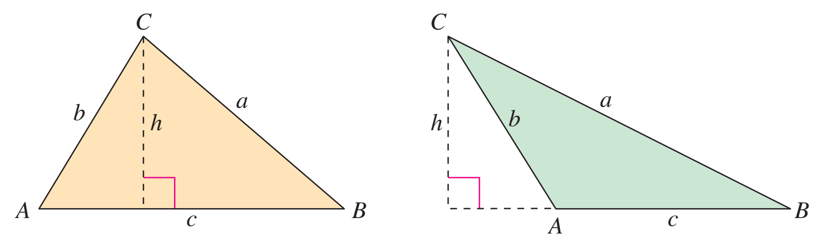
Area of an Oblique Triangle

The procedure used to prove the Law of Sines leads to a simple formula for the area of an oblique triangle. Referring to Figure 6.7, note that each triangle has a height of $h = b \sin A$. Consequently, the area of each triangle is

$$\text{Area} = \frac{1}{2}(\text{base})(\text{height}) = \frac{1}{2}(c)(b \sin A) = \frac{1}{2}bc \sin A.$$

By similar arguments, you can develop the formulas

$$\text{Area} = \frac{1}{2}ab \sin C = \frac{1}{2}ac \sin B.$$



A is acute *A is obtuse*

FIGURE 6.7

> **STUDY TIP**
>
> To see how to obtain the height of the obtuse triangle in Figure 6.7, notice the use of the reference angle $180° - A$ and the difference formula for sine, as follows.
>
> $h = b \sin(180° - A)$
>
> $= b(\sin 180° \cos A$
>
> $- \cos 180° \sin A)$
>
> $= b[0 \cdot \cos A - (-1) \cdot \sin A]$
>
> $= b \sin A$

Area of an Oblique Triangle

The area of any triangle is one-half the product of the lengths of two sides times the sine of their included angle. That is,

$$\text{Area} = \frac{1}{2}bc \sin A = \frac{1}{2}ab \sin C = \frac{1}{2}ac \sin B.$$

Note that if angle A is $90°$, the formula gives the area for a right triangle:

$$\text{Area} = \frac{1}{2}bc(\sin 90°) = \frac{1}{2}bc = \frac{1}{2}(\text{base})(\text{height}). \qquad \sin 90° = 1$$

Similar results are obtained for angles C and B equal to $90°$.

Activities

1. Use the given information to find (if possible) the remaining side and angles of the oblique triangle. If two solutions exist, find both.
   $A = 58°,\ a = 20,\ c = 10$
   Answer: $B = 97°,\ C = 25°,\ b = 23.4$
2. Use the given information to find (if possible) the remaining side and angles of the oblique triangle. If two solutions exist, find both.
   $B = 78°,\ b = 207,\ c = 210$
   Answer: Two solutions
   $A = 19.1°,\ a = 69.2,\ C = 82.9°$
   $A = 4.9°,\ a = 18.1,\ C = 97.1°$
3. Find the area of the triangle with $B = 120°$, $a = 32$, and $c = 50$.
   Answer: Area = 692.8 square units

Example 6 Finding the Area of a Triangular Lot

Find the area of a triangular lot having two sides of lengths 90 meters and 52 meters and an included angle of 102°.

Solution

Consider $a = 90$ meters, $b = 52$ meters, and angle $C = 102°$, as shown in Figure 6.8. Then, the area of the triangle is

$$\text{Area} = \frac{1}{2}ab \sin C = \frac{1}{2}(90)(52)(\sin 102°) \approx 2289 \text{ square meters.}$$

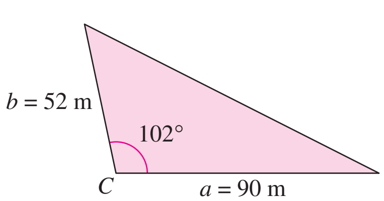


FIGURE 6.8

CHECKPOINT Now try Exercise 29.

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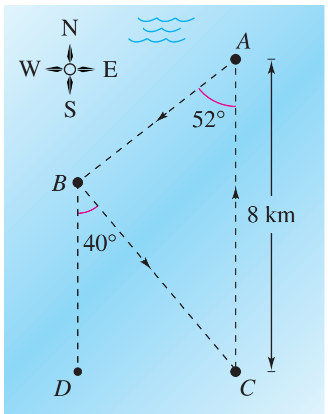


FIGURE 6.9

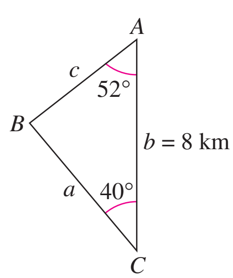


FIGURE 6.10

Application

Example 7 An Application of the Law of Sines

The course for a boat race starts at point A in Figure 6.9 and proceeds in the direction S 52° W to point B, then in the direction S 40° E to point C, and finally back to A. Point C lies 8 kilometers directly south of point A. Approximate the total distance of the race course.

Solution

Because lines BD and AC are parallel, it follows that $\angle BCA \cong \angle DBC$. Consequently, triangle ABC has the measures shown in Figure 6.10. For angle B, you have $B = 180° - 52° - 40° = 88°$. Using the Law of Sines

$$\frac{a}{\sin 52°} = \frac{b}{\sin 88°} = \frac{c}{\sin 40°}$$

you can let $b = 8$ and obtain

$$a = \frac{8}{\sin 88°}(\sin 52°) \approx 6.308$$

and

$$c = \frac{8}{\sin 88°}(\sin 40°) \approx 5.145.$$

The total length of the course is approximately

$$\text{Length} \approx 8 + 6.308 + 5.145$$

$$= 19.453 \text{ kilometers.}$$

CHECKPOINT Now try Exercise 39.

Alternative Writing About Mathematics: Error Analysis

You are a math instructor, and one of your students hands in the following solution. Discuss what is wrong with your student's solution. How could you help the student avoid making a similar mistake in the future? Use a diagram to illustrate your explanation.

Find side c in a triangle that has $a = 5.8$, $b = 7$, and $C = 82°$.

Student's Solution

Because this is an SSA situation, I can use the Law of Sines.

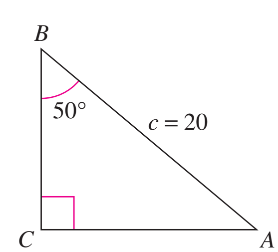
$$\frac{5.8}{7} = \frac{c}{\sin 82°}$$

$$(\sin 82°)\frac{5.8}{7} = 0.821$$

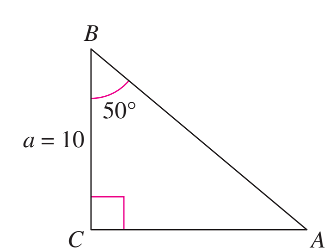
Writing About Mathematics

Using the Law of Sines In this section, you have been using the Law of Sines to solve *oblique* triangles. Can the Law of Sines also be used to solve a right triangle? If so, write a short paragraph explaining how to use the Law of Sines to solve each triangle. Is there an easier way to solve these triangles?

a. (AAS)



b. (ASA)



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6.1 Exercises

The *HM mathSpace®* CD-ROM and *Eduspace®* for this text contain step-by-step solutions to all odd-numbered exercises. They also provide Tutorial Exercises for additional help.

Exercises with no solution: 20, 21, 24
Exercises with two solutions: 6, 23

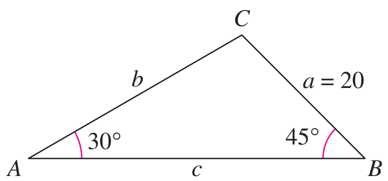
VOCABULARY CHECK: Fill in the blanks.

1. An ________ triangle is a triangle that has no right angle.
2. For triangle ABC, the Law of Sines is given by $\frac{a}{\sin A} =$ ________ $= \frac{c}{\sin C}$.
3. The area of an oblique triangle is given by $\frac{1}{2}bc \sin A = \frac{1}{2}ab \sin C =$ ________ .

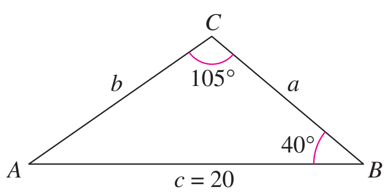
PREREQUISITE SKILLS REVIEW: Practice and review algebra skills needed for this section at **www.Eduspace.com.**

In Exercises 1–18, use the Law of Sines to solve the triangle. Round your answers to two decimal places.

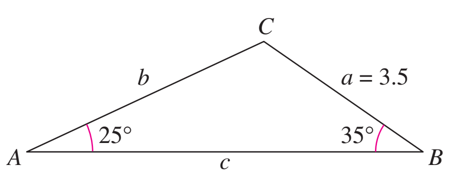
1.



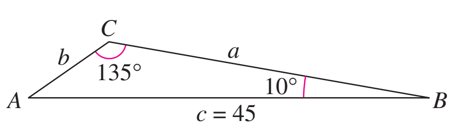
2.



3.



4.



5. $A = 36°, \quad a = 8, \quad b = 5$
6. $A = 60°, \quad a = 9, \quad c = 10$
7. $A = 102.4°, \quad C = 16.7°, \quad a = 21.6$
8. $A = 24.3°, \quad C = 54.6°, \quad c = 2.68$
9. $A = 83° 20', \quad C = 54.6°, \quad c = 18.1$
10. $A = 5° 40', \quad B = 8° 15', \quad b = 4.8$
11. $B = 15° 30', \quad a = 4.5, \quad b = 6.8$
12. $B = 2° 45', \quad b = 6.2, \quad c = 5.8$
13. $C = 145°, \quad b = 4, \quad c = 14$
14. $A = 100°, \quad a = 125, \quad c = 10$
15. $A = 110° 15', \quad a = 48, \quad b = 16$
16. $C = 85° 20', \quad a = 35, \quad c = 50$
17. $A = 55°, \quad B = 42°, \quad c = \frac{3}{4}$
18. $B = 28°, \quad C = 104°, \quad a = 3\frac{5}{8}$

In Exercises 19–24, use the Law of Sines to solve (if possible) the triangle. If two solutions exist, find both. Round your answers to two decimal places.

19. $A = 110°, \quad a = 125, \quad b = 100$
20. $A = 110°, \quad a = 125, \quad b = 200$
21. $A = 76°, \quad a = 18, \quad b = 20$
22. $A = 76°, \quad a = 34, \quad b = 21$
23. $A = 58°, \quad a = 11.4, \quad b = 12.8$
24. $A = 58°, \quad a = 4.5, \quad b = 12.8$

In Exercises 25–28, find values for *b* such that the triangle has (a) one solution, (b) two solutions, and (c) no solution.

25. $A = 36°, \quad a = 5$
26. $A = 60°, \quad a = 10$
27. $A = 10°, \quad a = 10.8$
28. $A = 88°, \quad a = 315.6$

In Exercises 29–34, find the area of the triangle having the indicated angle and sides.

29. $C = 120°, \quad a = 4, \quad b = 6$
30. $B = 130°, \quad a = 62, \quad c = 20$
31. $A = 43° 45', \quad b = 57, \quad c = 85$
32. $A = 5° 15', \quad b = 4.5, \quad c = 22$
33. $B = 72° 30', \quad a = 105, \quad c = 64$
34. $C = 84° 30', \quad a = 16, \quad b = 20$

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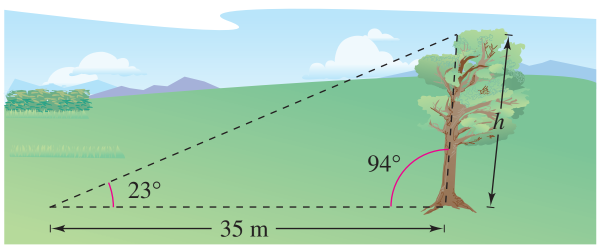
35. ***Height*** Because of prevailing winds, a tree grew so that it was leaning 4° from the vertical. At a point 35 meters from the tree, the angle of elevation to the top of the tree is 23° (see figure). Find the height h of the tree.



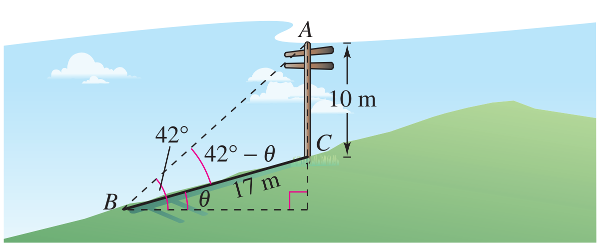
36. ***Height*** A flagpole at a right angle to the horizontal is located on a slope that makes an angle of 12° with the horizontal. The flagpole's shadow is 16 meters long and points directly up the slope. The angle of elevation from the tip of the shadow to the sun is 20°.

(a) Draw a triangle that represents the problem. Show the known quantities on the triangle and use a variable to indicate the height of the flagpole.

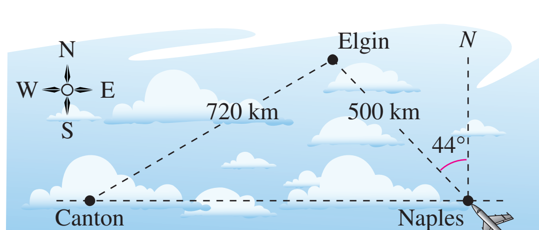
(b) Write an equation involving the unknown quantity.

(c) Find the height of the flagpole.

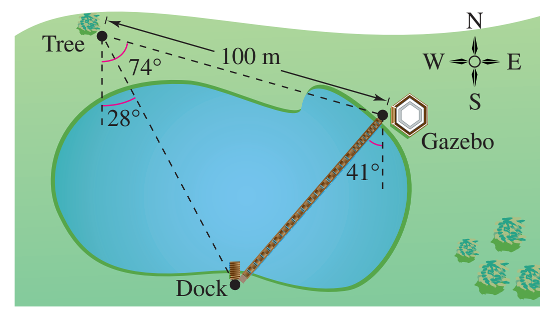
37. ***Angle of Elevation*** A 10-meter telephone pole casts a 17-meter shadow directly down a slope when the angle of elevation of the sun is 42° (see figure). Find θ, the angle of elevation of the ground.



38. ***Flight Path*** A plane flies 500 kilometers with a bearing of 316° from Naples to Elgin (see figure). The plane then flies 720 kilometers from Elgin to Canton. Find the bearing of the flight from Elgin to Canton.



39. ***Bridge Design*** A bridge is to be built across a small lake from a gazebo to a dock (see figure). The bearing from the gazebo to the dock is S 41° W. From a tree 100 meters from the gazebo, the bearings to the gazebo and the dock are S 74° E and S 28° E, respectively. Find the distance from the gazebo to the dock.



40. ***Railroad Track Design*** The circular arc of a railroad curve has a chord of length 3000 feet and a central angle of 40°.

(a) Draw a diagram that visually represents the problem. Show the known quantities on the diagram and use the variables r and s to represent the radius of the arc and the length of the arc, respectively.

(b) Find the radius r of the circular arc.

(c) Find the length s of the circular arc.

41. ***Glide Path*** A pilot has just started on the glide path for landing at an airport with a runway of length 9000 feet. The angles of depression from the plane to the ends of the runway are 17.5° and 18.8°.

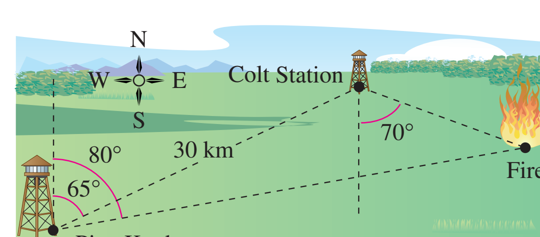
(a) Draw a diagram that visually represents the problem.

(b) Find the air distance the plane must travel until touching down on the near end of the runway.

(c) Find the ground distance the plane must travel until touching down.

(d) Find the altitude of the plane when the pilot begins the descent.

42. ***Locating a Fire*** The bearing from the Pine Knob fire tower to the Colt Station fire tower is N 65° E, and the two towers are 30 kilometers apart. A fire spotted by rangers in each tower has a bearing of N 80° E from Pine Knob and S 70° E from Colt Station (see figure). Find the distance of the fire from each tower.



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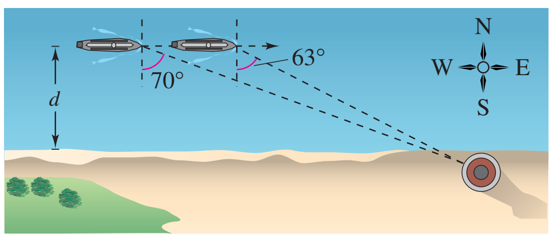
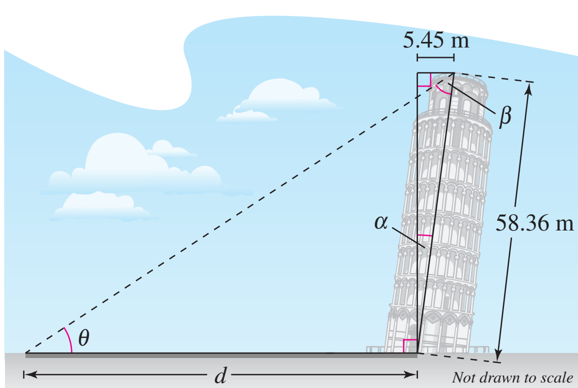
43. ***Distance*** A boat is sailing due east parallel to the shoreline at a speed of 10 miles per hour. At a given time, the bearing to the lighthouse is S 70° E, and 15 minutes later the bearing is S 63° E (see figure). The lighthouse is located at the shoreline. What is the distance from the boat to the shoreline?



Model It

44. ***Shadow Length*** The Leaning Tower of Pisa in Italy is characterized by its tilt. The tower leans because it was built on a layer of unstable soil—clay, sand, and water. The tower is approximately 58.36 meters tall from its foundation (see figure). The top of the tower leans about 5.45 meters off center.



(a) Find the angle of lean α of the tower.

(b) Write β as a function of d and θ, where θ is the angle of elevation to the sun.

(c) Use the Law of Sines to write an equation for the length d of the shadow cast by the tower.

(d) Use a graphing utility to complete the table.

θ	10°	20°	30°	40°	50°	60°
d						

Synthesis

True or False? **In Exercises 45 and 46, determine whether the statement is true or false. Justify your answer.**

45. If a triangle contains an obtuse angle, then it must be oblique.

46. Two angles and one side of a triangle do not necessarily determine a unique triangle.

47. ***Graphical and Numerical Analysis*** In the figure, α and β are positive angles.

(a) Write α as a function of β.

(b) Use a graphing utility to graph the function. Determine its domain and range.

(c) Use the result of part (a) to write c as a function of β.

(d) Use a graphing utility to graph the function in part (c). Determine its domain and range.

(e) Complete the table. What can you infer?

β	0.4	0.8	1.2	1.6	2.0	2.4	2.8
α							
c							

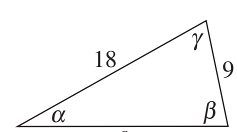


FIGURE FOR 47

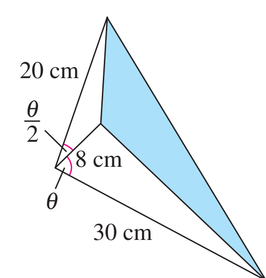


FIGURE FOR 48

48. ***Graphical Analysis***

(a) Write the area A of the shaded region in the figure as a function of θ.

(b) Use a graphing utility to graph the area function.

(c) Determine the domain of the area function. Explain how the area of the region and the domain of the function would change if the eight-centimeter line segment were decreased in length.

Skills Review

In Exercises 49–52, use the fundamental trigonometric identities to simplify the expression.

49. $\sin x \cot x$

50. $\tan x \cos x \sec x$

51. $1 - \sin^2\left(\frac{\pi}{2} - x\right)$

52. $1 + \cot^2\left(\frac{\pi}{2} - x\right)$