10.3 Ellipses

What you should learn

- Write equations of ellipses in standard form and graph ellipses.
- Use properties of ellipses to model and solve real-life problems.
- Find eccentricities of ellipses.

Why you should learn it

Ellipses can be used to model and solve many types of real-life problems. For instance, in Exercise 59 on page 751, an ellipse is used to model the orbit of Halley's comet.

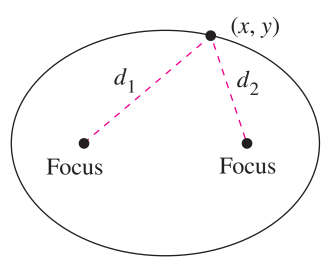
Harvard College Observatory/ SPL/Photo Researchers, Inc.

Introduction

The second type of conic is called an **ellipse,** and is defined as follows.

Definition of Ellipse

An **ellipse** is the set of all points (x, y) in a plane, the sum of whose distances from two distinct fixed points **(foci)** is constant. See Figure 10.18.



$d_1 + d_2$ is constant.
FIGURE 10.18

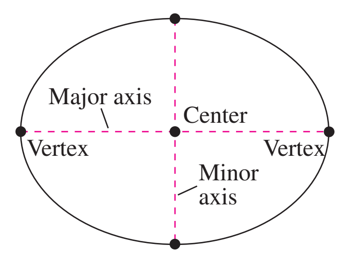


FIGURE 10.19

The line through the foci intersects the ellipse at two points called **vertices.** The chord joining the vertices is the **major axis,** and its midpoint is the **center** of the ellipse. The chord perpendicular to the major axis at the center is the **minor axis** of the ellipse. See Figure 10.19.

You can visualize the definition of an ellipse by imagining two thumbtacks placed at the foci, as shown in Figure 10.20. If the ends of a fixed length of string are fastened to the thumbtacks and the string is *drawn taut* with a pencil, the path traced by the pencil will be an ellipse.

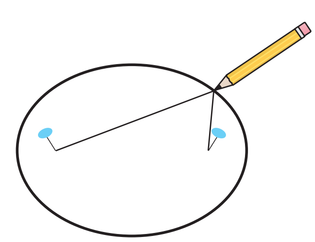


FIGURE 10.20

To derive the standard form of the equation of an ellipse, consider the ellipse in Figure 10.21 with the following points: center, (h, k); vertices, $(h \pm a, k)$; foci, $(h \pm c, k)$. Note that the center is the midpoint of the segment joining the foci.

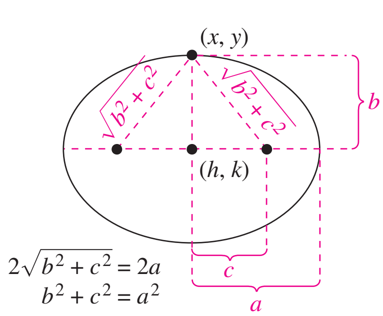


FIGURE 10.21

When discussing ellipses, you might also choose to discuss the latera recta as background for Exercises 62–66.

The sum of the distances from any point on the ellipse to the two foci is constant. Using a vertex point, this constant sum is

$$(a + c) + (a - c) = 2a \qquad \text{Length of major axis}$$

or simply the length of the major axis. Now, if you let (x, y) be *any* point on the ellipse, the sum of the distances between (x, y) and the two foci must also be $2a$. That is,

$$\sqrt{[x - (h - c)]^2 + (y - k)^2} + \sqrt{[x - (h + c)]^2 + (y - k)^2} = 2a.$$

Finally, in Figure 10.21, you can see that $b^2 = a^2 - c^2$, which implies that the equation of the ellipse is

$$b^2(x - h)^2 + a^2(y - k)^2 = a^2b^2$$

$$\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1.$$

You would obtain a similar equation in the derivation by starting with a vertical major axis. Both results are summarized as follows.

STUDY TIP

Consider the equation of the ellipse

$$\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1.$$

If you let $a = b$, then the equation can be rewritten as

$$(x - h)^2 + (y - k)^2 = a^2$$

which is the standard form of the equation of a circle with radius $r = a$ (see Section 1.2). Geometrically, when $a = b$ for an ellipse, the major and minor axes are of equal length, and so the graph is a circle.

Standard Equation of an Ellipse

The **standard form of the equation of an ellipse,** with center (h, k) and major and minor axes of lengths $2a$ and $2b$, respectively, where $0 < b < a$, is

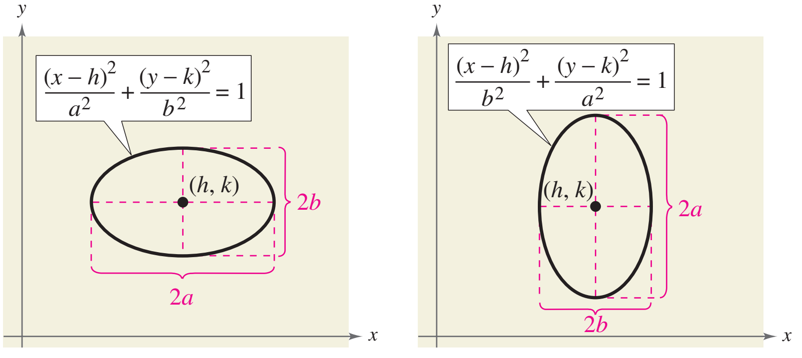
$$\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1 \qquad \text{Major axis is horizontal.}$$

$$\frac{(x - h)^2}{b^2} + \frac{(y - k)^2}{a^2} = 1. \qquad \text{Major axis is vertical.}$$

The foci lie on the major axis, c units from the center, with $c^2 = a^2 - b^2$. If the center is at the origin $(0, 0)$, the equation takes one of the following forms.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{Major axis is horizontal.} \qquad \frac{x^2}{b^2} + \frac{y^2}{a^2} = 1 \quad \text{Major axis is vertical.}$$

Figure 10.22 shows both the horizontal and vertical orientations for an ellipse.



Major axis is horizontal. *Major axis is vertical.*

FIGURE 10.22

Example 1 Finding the Standard Equation of an Ellipse

Find the standard form of the equation of the ellipse having foci at $(0, 1)$ and $(4, 1)$ and a major axis of length 6, as shown in Figure 10.23.

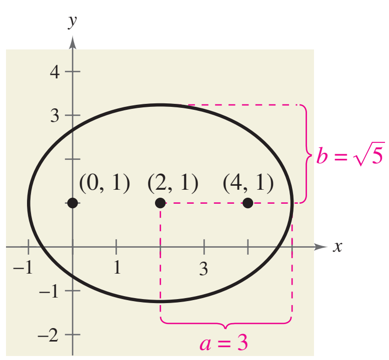


FIGURE 10.23

Solution

Because the foci occur at $(0, 1)$ and $(4, 1)$, the center of the ellipse is $(2, 1)$ and the distance from the center to one of the foci is $c = 2$. Because $2a = 6$, you know that $a = 3$. Now, from $c^2 = a^2 - b^2$, you have

$$b = \sqrt{a^2 - c^2} = \sqrt{3^2 - 2^2} = \sqrt{5}.$$

Because the major axis is horizontal, the standard equation is

$$\frac{(x-2)^2}{3^2} + \frac{(y-1)^2}{(\sqrt{5})^2} = 1.$$

This equation simplifies to

$$\frac{(x-2)^2}{9} + \frac{(y-1)^2}{5} = 1.$$

CHECKPOINT Now try Exercise 49.

Remind your students that completing the square must be performed twice to write the equation of the ellipse in standard form in Example 2.

Example 2 Sketching an Ellipse

Sketch the ellipse given by $x^2 + 4y^2 + 6x - 8y + 9 = 0$.

Solution

Begin by writing the original equation in standard form. In the fourth step, note that 9 and 4 are added to *both* sides of the equation when completing the squares.

$$x^2 + 4y^2 + 6x - 8y + 9 = 0 \quad \text{Write original equation.}$$

$$(x^2 + 6x + \square) + (4y^2 - 8y + \square) = -9 \quad \text{Group terms.}$$

$$(x^2 + 6x + \square) + 4(y^2 - 2y + \square) = -9 \quad \text{Factor 4 out of } y\text{-terms.}$$

$$(x^2 + 6x + 9) + 4(y^2 - 2y + 1) = -9 + 9 + 4(1)$$

$$(x + 3)^2 + 4(y - 1)^2 = 4 \quad \text{Write in completed square form.}$$

$$\frac{(x+3)^2}{4} + \frac{(y-1)^2}{1} = 1 \quad \text{Divide each side by 4.}$$

$$\frac{(x+3)^2}{2^2} + \frac{(y-1)^2}{1^2} = 1 \quad \text{Write in standard form.}$$

From this standard form, it follows that the center is $(h, k) = (-3, 1)$. Because the denominator of the x-term is $a^2 = 2^2$, the endpoints of the major axis lie two units to the right and left of the center. Similarly, because the denominator of the y-term is $b^2 = 1^2$, the endpoints of the minor axis lie one unit up and down from the center. Now, from $c^2 = a^2 - b^2$, you have $c = \sqrt{2^2 - 1^2} = \sqrt{3}$. So, the foci of the ellipse are $(-3 - \sqrt{3}, 1)$ and $(-3 + \sqrt{3}, 1)$. The ellipse is shown in Figure 10.24.

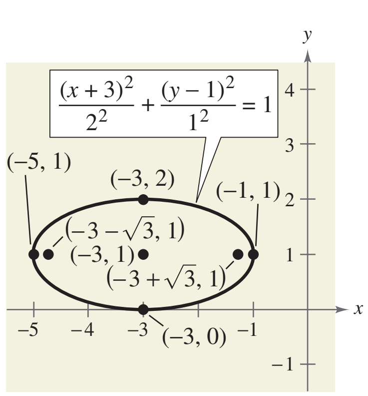


FIGURE 10.24

CHECKPOINT Now try Exercise 25.

Example 3 Analyzing an Ellipse

Find the center, vertices, and foci of the ellipse $4x^2 + y^2 - 8x + 4y - 8 = 0$.

Solution

By completing the square, you can write the original equation in standard form.

$$4x^2 + y^2 - 8x + 4y - 8 = 0 \quad \text{Write original equation.}$$

$$(4x^2 - 8x + \square) + (y^2 + 4y + \square) = 8 \quad \text{Group terms.}$$

$$4(x^2 - 2x + \square) + (y^2 + 4y + \square) = 8 \quad \text{Factor 4 out of } x\text{-terms.}$$

$$4(x^2 - 2x + 1) + (y^2 + 4y + 4) = 8 + 4(1) + 4$$

$$4(x - 1)^2 + (y + 2)^2 = 16 \quad \text{Write in completed square form.}$$

$$\frac{(x-1)^2}{4} + \frac{(y+2)^2}{16} = 1 \quad \text{Divide each side by 16.}$$

$$\frac{(x-1)^2}{2^2} + \frac{(y+2)^2}{4^2} = 1 \quad \text{Write in standard form.}$$

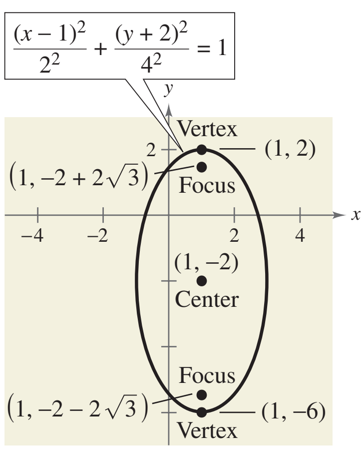


FIGURE 10.25

The major axis is vertical, where $h = 1$, $k = -2$, $a = 4$, $b = 2$, and

$$c = \sqrt{a^2 - b^2} = \sqrt{16 - 4} = \sqrt{12} = 2\sqrt{3}.$$

So, you have the following.

Center: $(1, -2)$ Vertices: $(1, -6)$, $(1, 2)$ Foci: $(1, -2 - 2\sqrt{3})$, $(1, -2 + 2\sqrt{3})$

The graph of the ellipse is shown in Figure 10.25.

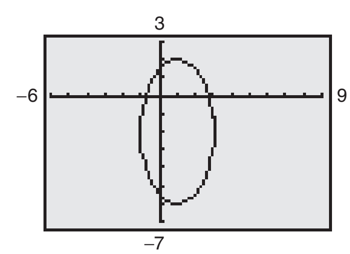
CHECKPOINT Now try Exercise 29.

Technology

You can use a graphing utility to graph an ellipse by graphing the upper and lower portions in the same viewing window. For instance, to graph the ellipse in Example 3, first solve for y to get

$$y_1 = -2 + 4\sqrt{1 - \frac{(x-1)^2}{4}} \quad \text{and} \quad y_2 = -2 - 4\sqrt{1 - \frac{(x-1)^2}{4}}.$$

Use a viewing window in which $-6 \le x \le 9$ and $-7 \le y \le 3$. You should obtain the graph shown below.



Application

Ellipses have many practical and aesthetic uses. For instance, machine gears, supporting arches, and acoustic designs often involve elliptical shapes. The orbits of satellites and planets are also ellipses. Example 4 investigates the elliptical orbit of the moon about Earth.

Example 4 An Application Involving an Elliptical Orbit

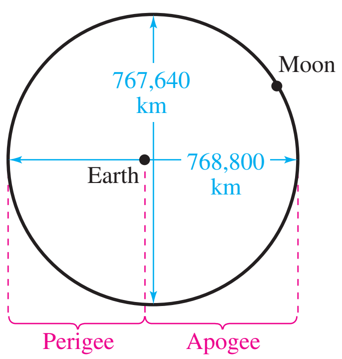


FIGURE 10.26

The moon travels about Earth in an elliptical orbit with Earth at one focus, as shown in Figure 10.26. The major and minor axes of the orbit have lengths of 768,800 kilometers and 767,640 kilometers, respectively. Find the greatest and smallest distances (the *apogee* and *perigee*), respectively from Earth's center to the moon's center.

Solution

Because $2a = 768{,}800$ and $2b = 767{,}640$, you have

$$a = 384{,}400 \text{ and } b = 383{,}820$$

which implies that

$$\begin{aligned} c &= \sqrt{a^2 - b^2} \\ &= \sqrt{384{,}400^2 - 383{,}820^2} \\ &\approx 21{,}108. \end{aligned}$$

So, the greatest distance between the center of Earth and the center of the moon is

$$a + c \approx 384{,}400 + 21{,}108 = 405{,}508 \text{ kilometers}$$

and the smallest distance is

$$a - c \approx 384{,}400 - 21{,}108 = 363{,}292 \text{ kilometers.}$$

CHECKPOINT Now try Exercise 59.

STUDY TIP

Note in Example 4 and Figure 10.26 that Earth *is not* the center of the moon's orbit.

Eccentricity

One of the reasons it was difficult for early astronomers to detect that the orbits of the planets are ellipses is that the foci of the planetary orbits are relatively close to their centers, and so the orbits are nearly circular. To measure the ovalness of an ellipse, you can use the concept of **eccentricity.**

Ask students to make a conjecture about the eccentricity of a circle before going further.

Definition of Eccentricity

The **eccentricity** e of an ellipse is given by the ratio

$$e = \frac{c}{a}.$$

Note that $0 < e < 1$ for *every* ellipse.

Activities

1. Find the center, foci, vertices, and eccentricity of the ellipse

$$\frac{(x-2)^2}{25} + \frac{(y+1)^2}{9} = 1.$$

Answer: Center $(2, -1)$; Foci $(-2, -1)$, $(6, -1)$; Vertices $(-3, -1)$, $(7, -1)$; $e = \frac{4}{5}$

2. Rewrite the equation of the ellipse in standard form: $9x^2 + 5y^2 + 36x - 30y + 36 = 0$.

Answer: $\frac{(x+2)^2}{(\sqrt{5})^2} + \frac{(y-3)^2}{3^2} = 1$

To see how this ratio is used to describe the shape of an ellipse, note that because the foci of an ellipse are located along the major axis between the vertices and the center, it follows that

$$0 < c < a.$$

For an ellipse that is nearly circular, the foci are close to the center and the ratio c/a is small, as shown in Figure 10.27. On the other hand, for an elongated ellipse, the foci are close to the vertices, and the ratio c/a is close to 1, as shown in Figure 10.28.

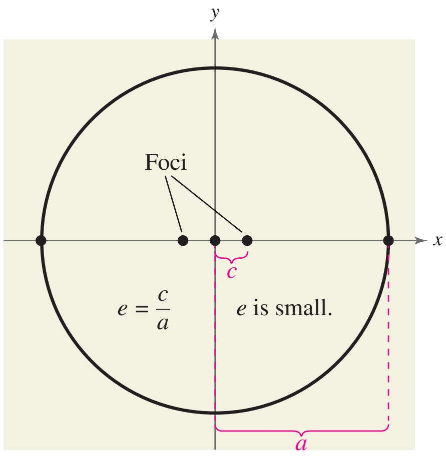


FIGURE 10.27

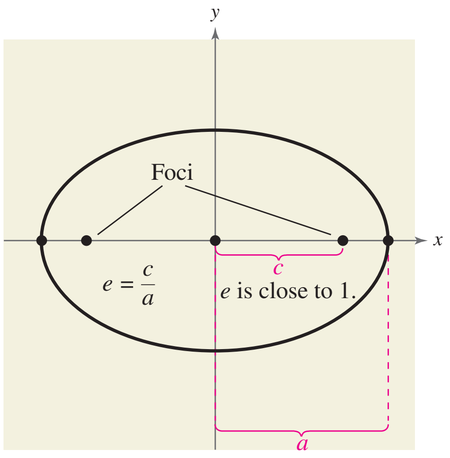


FIGURE 10.28

NASA

The time it takes Saturn to orbit the sun is equal to 29.4 Earth years.

The orbit of the moon has an eccentricity of $e \approx 0.0549$, and the eccentricities of the nine planetary orbits are as follows.

Mercury:	$e \approx 0.2056$	Saturn:	$e \approx 0.0542$
Venus:	$e \approx 0.0068$	Uranus:	$e \approx 0.0472$
Earth:	$e \approx 0.0167$	Neptune:	$e \approx 0.0086$
Mars:	$e \approx 0.0934$	Pluto:	$e \approx 0.2488$
Jupiter:	$e \approx 0.0484$		

WRITING ABOUT MATHEMATICS

Ellipses and Circles

a. Show that the equation of an ellipse can be written as

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{a^2(1-e^2)} = 1.$$

b. For the equation in part (a), let $a = 4$, $h = 1$, and $k = 2$, and use a graphing utility to graph the ellipse for $e = 0.95$, $e = 0.75$, $e = 0.5$, $e = 0.25$, and $e = 0.1$. Discuss the changes in the shape of the ellipse as e approaches 0.

c. Make a conjecture about the shape of the graph in part (b) when $e = 0$. What is the equation of this ellipse? What is another name for an ellipse with an eccentricity of 0?

10.3 Exercises

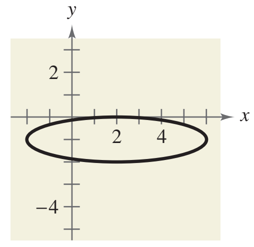
VOCABULARY CHECK: Fill in the blanks.

1. An ________ is the set of all points (x, y) in a plane, the sum of whose distances from two distinct fixed points, called ________, is constant.
2. The chord joining the vertices of an ellipse is called the ________ ________, and its midpoint is the ________ of the ellipse.
3. The chord perpendicular to the major axis at the center of the ellipse is called the ________ ________ of the ellipse.
4. The concept of ________ is used to measure the ovalness of an ellipse.

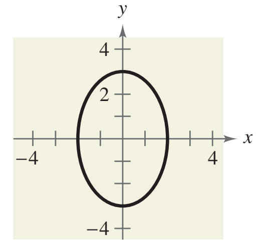
PREREQUISITE SKILLS REVIEW: Practice and review algebra skills needed for this section at **www.Eduspace.com.**

In Exercises 1–6, match the equation with its graph. [The graphs are labeled (a), (b), (c), (d), (e), and (f).]

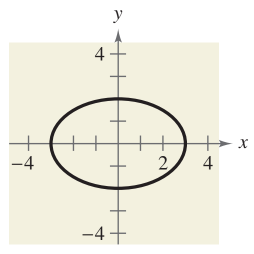
(a)



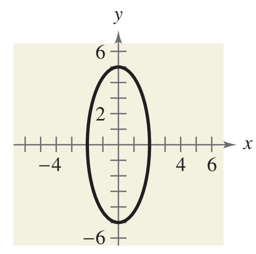
(b)



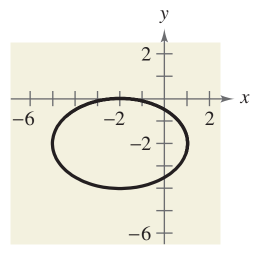
(c)



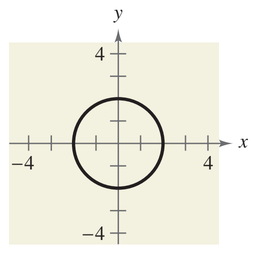
(d)



(e)



(f)



1. $\dfrac{x^2}{4} + \dfrac{y^2}{9} = 1$
2. $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$
3. $\dfrac{x^2}{4} + \dfrac{y^2}{25} = 1$
4. $\dfrac{x^2}{4} + \dfrac{y^2}{4} = 1$
5. $\dfrac{(x-2)^2}{16} + (y+1)^2 = 1$
6. $\dfrac{(x+2)^2}{9} + \dfrac{(y+2)^2}{4} = 1$

In Exercises 7–30, identify the conic as a circle or an ellipse. Then find the center, radius, vertices, foci, and eccentricity of the conic (if applicable), and sketch its graph.

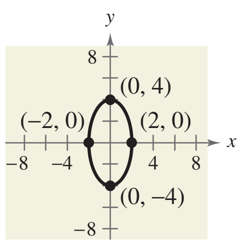
7. $\dfrac{x^2}{25} + \dfrac{y^2}{16} = 1$
8. $\dfrac{x^2}{81} + \dfrac{y^2}{144} = 1$
9. $\dfrac{x^2}{25} + \dfrac{y^2}{25} = 1$
10. $\dfrac{x^2}{9} + \dfrac{y^2}{9} = 1$
11. $\dfrac{x^2}{5} + \dfrac{y^2}{9} = 1$
12. $\dfrac{x^2}{64} + \dfrac{y^2}{28} = 1$
13. $\dfrac{(x+3)^2}{16} + \dfrac{(y-5)^2}{25} = 1$
14. $\dfrac{(x-4)^2}{12} + \dfrac{(y+3)^2}{16} = 1$
15. $\dfrac{x^2}{4/9} + \dfrac{(y+1)^2}{4/9} = 1$
16. $\dfrac{(x+5)^2}{9/4} + (y-1)^2 = 1$
17. $(x+2)^2 + \dfrac{(y+4)^2}{1/4} = 1$
18. $\dfrac{(x-3)^2}{25/4} + \dfrac{(y-1)^2}{25/4} = 1$
19. $9x^2 + 4y^2 + 36x - 24y + 36 = 0$
20. $9x^2 + 4y^2 - 54x + 40y + 37 = 0$
21. $x^2 + y^2 - 2x + 4y - 31 = 0$
22. $x^2 + 5y^2 - 8x - 30y - 39 = 0$
23. $3x^2 + y^2 + 18x - 2y - 8 = 0$
24. $6x^2 + 2y^2 + 18x - 10y + 2 = 0$
25. $x^2 + 4y^2 - 6x + 20y - 2 = 0$
26. $x^2 + y^2 - 4x + 6y - 3 = 0$
27. $9x^2 + 9y^2 + 18x - 18y + 14 = 0$
28. $16x^2 + 25y^2 - 32x + 50y + 16 = 0$
29. $9x^2 + 25y^2 - 36x - 50y + 60 = 0$
30. $16x^2 + 16y^2 - 64x + 32y + 55 = 0$

In Exercises 31–34, use a graphing utility to graph the ellipse. Find the center, foci, and vertices. (Recall that it may be necessary to solve the equation for *y* and obtain two equations.)

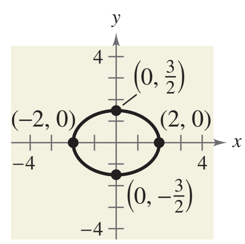
31. $5x^2 + 3y^2 = 15$
32. $3x^2 + 4y^2 = 12$
33. $12x^2 + 20y^2 - 12x + 40y - 37 = 0$
34. $36x^2 + 9y^2 + 48x - 36y - 72 = 0$

In Exercises 35–42, find the standard form of the equation of the ellipse with the given characteristics and center at the origin.

35.



36.



37. Vertices: $(\pm 6, 0)$; foci: $(\pm 2, 0)$

38. Vertices: $(0, \pm 8)$; foci: $(0, \pm 4)$

39. Foci: $(\pm 5, 0)$; major axis of length 12

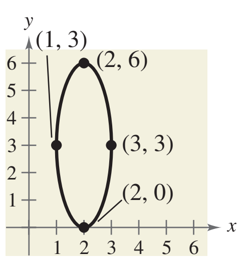
40. Foci: $(\pm 2, 0)$; major axis of length 8

41. Vertices: $(0, \pm 5)$; passes through the point $(4, 2)$

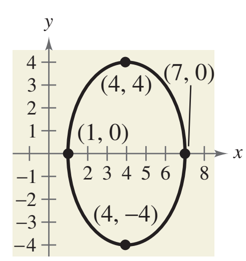
42. Major axis vertical; passes through the points $(0, 4)$ and $(2, 0)$

In Exercises 43–54, find the standard form of the equation of the ellipse with the given characteristics.

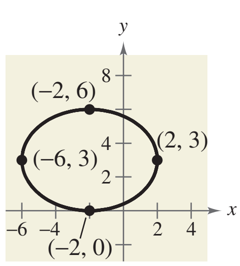
43.



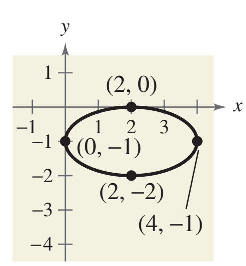
44.



45.



46.



47. Vertices: $(0, 4), (4, 4)$; minor axis of length 2

48. Foci: $(0, 0), (4, 0)$; major axis of length 8

49. Foci: $(0, 0), (0, 8)$; major axis of length 16

50. Center: $(2, -1)$; vertex: $\left(2, \frac{1}{2}\right)$; minor axis of length 2

51. Center: $(0, 4)$; $a = 2c$; vertices: $(-4, 4), (4, 4)$

52. Center: $(3, 2)$; $a = 3c$; foci: $(1, 2), (5, 2)$

53. Vertices: $(0, 2), (4, 2)$; endpoints of the minor axis: $(2, 3), (2, 1)$

54. Vertices: $(5, 0), (5, 12)$; endpoints of the minor axis: $(1, 6), (9, 6)$

55. Find an equation of the ellipse with vertices $(\pm 5, 0)$ and eccentricity $e = \frac{3}{5}$.

56. Find an equation of the ellipse with vertices $(0, \pm 8)$ and eccentricity $e = \frac{1}{2}$.

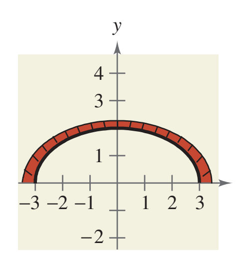
57. ***Architecture*** A semielliptical arch over a tunnel for a one-way road through a mountain has a major axis of 50 feet and a height at the center of 10 feet.

(a) Draw a rectangular coordinate system on a sketch of the tunnel with the center of the road entering the tunnel at the origin. Identify the coordinates of the known points.

(b) Find an equation of the semielliptical arch over the tunnel.

(c) You are driving a moving truck that has a width of 8 feet and a height of 9 feet. Will the moving truck clear the opening of the arch?

58. ***Architecture*** A fireplace arch is to be constructed in the shape of a semiellipse. The opening is to have a height of 2 feet at the center and a width of 6 feet along the base (see figure). The contractor draws the outline of the ellipse using tacks as described at the beginning of this section. Give the required positions of the tacks and the length of the string.



Model It

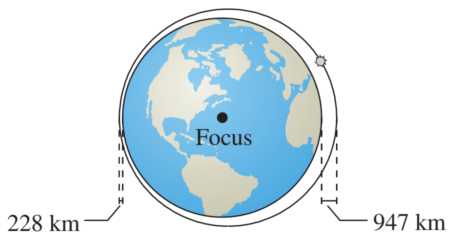
59. ***Comet Orbit*** Halley's comet has an elliptical orbit, with the sun at one focus. The eccentricity of the orbit is approximately 0.967. The length of the major axis of the orbit is approximately 35.88 astronomical units. (An astronomical unit is about 93 million miles.)

(a) Find an equation of the orbit. Place the center of the orbit at the origin, and place the major axis on the x-axis.

(b) Use a graphing utility to graph the equation of the orbit.

(c) Find the greatest (aphelion) and smallest (perihelion) distances from the sun's center to the comet's center.

60. ***Satellite Orbit*** The first artificial satellite to orbit Earth was Sputnik I (launched by the former Soviet Union in 1957). Its highest point above Earth's surface was 947 kilometers, and its lowest point was 228 kilometers (see figure). The center of Earth was the focus of the elliptical orbit, and the radius of Earth is 6378 kilometers. Find the eccentricity of the orbit.



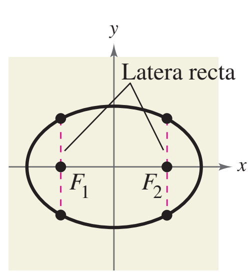
61. ***Motion of a Pendulum*** The relation between the velocity y (in radians per second) of a pendulum and its angular displacement θ from the vertical can be modeled by a semiellipse. A 12-centimeter pendulum crests $(y = 0)$ when the angular displacement is -0.2 radian and 0.2 radian. When the pendulum is at equilibrium $(\theta = 0)$, the velocity is -1.6 radians per second.

(a) Find an equation that models the motion of the pendulum. Place the center at the origin.

(b) Graph the equation from part (a).

(c) Which half of the ellipse models the motion of the pendulum?

62. ***Geometry*** A line segment through a focus of an ellipse with endpoints on the ellipse and perpendicular to the major axis is called a **latus rectum** of the ellipse. Therefore, an ellipse has two latera recta. Knowing the length of the latera recta is helpful in sketching an ellipse because it yields other points on the curve (see figure). Show that the length of each latus rectum is $2b^2/a$.



In Exercises 63–66, sketch the graph of the ellipse, using latera recta (see Exercise 62).

63. $\dfrac{x^2}{9} + \dfrac{y^2}{16} = 1$

64. $\dfrac{x^2}{4} + \dfrac{y^2}{1} = 1$

65. $5x^2 + 3y^2 = 15$

66. $9x^2 + 4y^2 = 36$

Synthesis

True or False? **In Exercises 67 and 68, determine whether the statement is true or false. Justify your answer.**

67. The graph of $x^2 + 4y^4 - 4 = 0$ is an ellipse.

68. It is easier to distinguish the graph of an ellipse from the graph of a circle if the eccentricity of the ellipse is large (close to 1).

69. ***Exploration*** Consider the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad a + b = 20.$$

(a) The area of the ellipse is given by $A = \pi ab$. Write the area of the ellipse as a function of a.

(b) Find the equation of an ellipse with an area of 264 square centimeters.

(c) Complete the table using your equation from part (a), and make a conjecture about the shape of the ellipse with maximum area.

a	8	9	10	11	12	13
A						

(d) Use a graphing utility to graph the area function and use the graph to support your conjecture in part (c).

70. ***Think About It*** At the beginning of this section it was noted that an ellipse can be drawn using two thumbtacks, a string of fixed length (greater than the distance between the two tacks), and a pencil. If the ends of the string are fastened at the tacks and the string is drawn taut with a pencil, the path traced by the pencil is an ellipse.

(a) What is the length of the string in terms of a?

(b) Explain why the path is an ellipse.

Skills Review

In Exercises 71–74, determine whether the sequence is arithmetic, geometric, or neither.

71. 80, 40, 20, 10, 5, . . .

72. 66, 55, 44, 33, 22, . . .

73. $-\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \ldots$

74. $\frac{1}{4}, \frac{1}{2}, 1, 2, 4, \ldots$

In Exercises 75–78, find the sum.

75. $\displaystyle\sum_{n=0}^{6} (-3)^n$

76. $\displaystyle\sum_{n=0}^{6} 3^n$

77. $\displaystyle\sum_{n=0}^{10} 5\left(\frac{4}{3}\right)^n$

78. $\displaystyle\sum_{n=1}^{10} 4\left(\frac{3}{4}\right)^{n-1}$