# 3.5 Exponential and Logarithmic Models

## What you should learn

- Recognize the five most common types of models involving exponential and logarithmic functions.
- Use exponential growth and decay functions to model and solve real-life problems.
- Use Gaussian functions to model and solve real-life problems.
- Use logistic growth functions to model and solve real-life problems.
- Use logarithmic functions to model and solve real-life problems.

## Why you should learn it

Exponential growth and decay models are often used to model the population of a country. For instance, in Exercise 36 on page 265, you will use exponential growth and decay models to compare the populations of several countries.


Alan Becker/Getty Images

## Introduction

The five most common types of mathematical models involving exponential functions and logarithmic functions are as follows.

1. **Exponential growth model:** $y = ae^{bx}, \quad b > 0$
2. **Exponential decay model:** $y = ae^{-bx}, \quad b > 0$
3. **Gaussian model:** $y = ae^{-(x-b)^2/c}$
4. **Logistic growth model:** $y = \dfrac{a}{1 + be^{-rx}}$
5. **Logarithmic models:** $y = a + b \ln x, \quad y = a + b \log x$

The basic shapes of the graphs of these functions are shown in Figure 3.29.

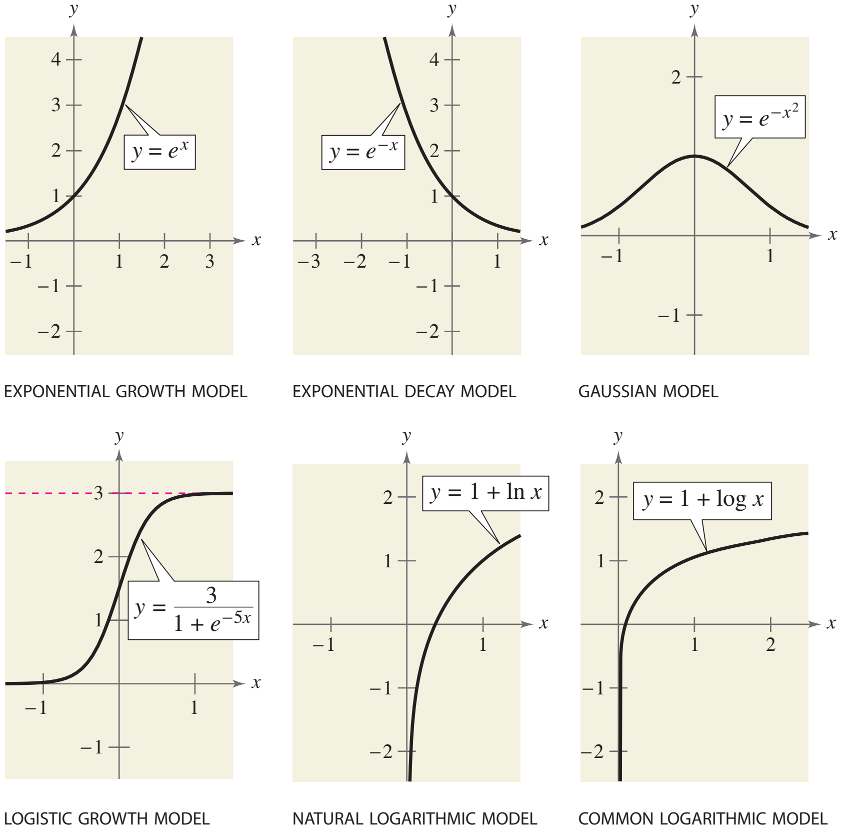



FIGURE 3.29

You can often gain quite a bit of insight into a situation modeled by an exponential or logarithmic function by identifying and interpreting the function's asymptotes. Use the graphs in Figure 3.29 to identify the asymptotes of the graph of each function.

This section shows students real-world applications of logarithmic and exponential functions.

## Exponential Growth and Decay

### Example 1 Digital Television

Estimates of the numbers (in millions) of U.S. households with digital television from 2003 through 2007 are shown in the table. The scatter plot of the data is shown in Figure 3.30. (Source: eMarketer)

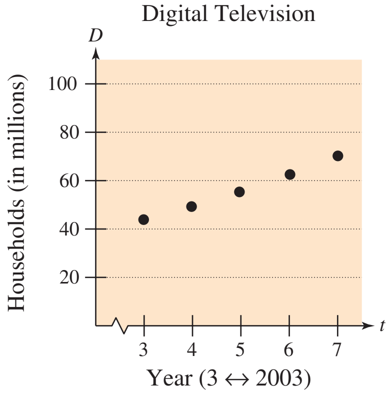



FIGURE 3.30

| Year | Households |
|---|---|
| 2003 | 44.2 |
| 2004 | 49.0 |
| 2005 | 55.5 |
| 2006 | 62.5 |
| 2007 | 70.3 |

An exponential growth model that approximates these data is given by

$$D = 30.92e^{0.1171t}, \quad 3 \le t \le 7$$

where $D$ is the number of households (in millions) and $t = 3$ represents 2003. Compare the values given by the model with the estimates shown in the table. According to this model, when will the number of U.S. households with digital television reach 100 million?

#### Solution

The following table compares the two sets of figures. The graph of the model and the original data are shown in Figure 3.31.

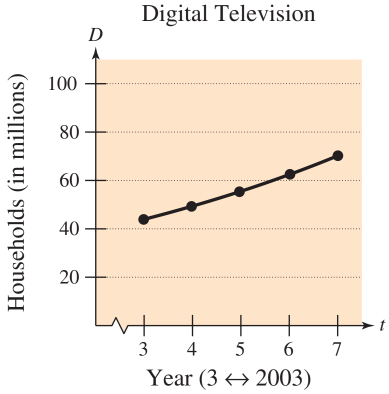



FIGURE 3.31

| Year | 2003 | 2004 | 2005 | 2006 | 2007 |
|---|---|---|---|---|---|
| Households | 44.2 | 49.0 | 55.5 | 62.5 | 70.3 |
| Model | 43.9 | 49.4 | 55.5 | 62.4 | 70.2 |

To find when the number of U.S. households with digital television will reach 100 million, let $D = 100$ in the model and solve for $t$.

$$30.92e^{0.1171t} = D \qquad \text{Write original model.}$$

$$30.92e^{0.1171t} = 100 \qquad \text{Let } D = 100.$$

$$e^{0.1171t} \approx 3.2342 \qquad \text{Divide each side by 30.92.}$$

$$\ln e^{0.1171t} \approx \ln 3.2342 \qquad \text{Take natural log of each side.}$$

$$0.1171t \approx 1.1738 \qquad \text{Inverse Property}$$

$$t \approx 10.0 \qquad \text{Divide each side by 0.1171.}$$

According to the model, the number of U.S. households with digital television will reach 100 million in 2010.

CHECKPOINT Now try Exercise 35.

**Technology**

Some graphing utilities have an *exponential regression* feature that can be used to find exponential models that represent data. If you have such a graphing utility, try using it to find an exponential model for the data given in Example 1. How does your model compare with the model given in Example 1?

In Example 1, you were given the exponential growth model. But suppose this model were not given; how could you find such a model? One technique for doing this is demonstrated in Example 2.

## Example 2 Modeling Population Growth

In a research experiment, a population of fruit flies is increasing according to the law of exponential growth. After 2 days there are 100 flies, and after 4 days there are 300 flies. How many flies will there be after 5 days?

### Solution

Let $y$ be the number of flies at time $t$. From the given information, you know that $y = 100$ when $t = 2$ and $y = 300$ when $t = 4$. Substituting this information into the model $y = ae^{bt}$ produces

$$100 = ae^{2b} \quad \text{and} \quad 300 = ae^{4b}.$$

To solve for $b$, solve for $a$ in the first equation.

$$100 = ae^{2b} \quad \Longrightarrow \quad a = \frac{100}{e^{2b}} \qquad \text{Solve for } a \text{ in the first equation.}$$

Then substitute the result into the second equation.

$$300 = ae^{4b} \qquad \text{Write second equation.}$$

$$300 = \left(\frac{100}{e^{2b}}\right)e^{4b} \qquad \text{Substitute } 100/e^{2b} \text{ for } a.$$

$$\frac{300}{100} = e^{2b} \qquad \text{Divide each side by 100.}$$

$$\ln 3 = 2b \qquad \text{Take natural log of each side.}$$

$$\frac{1}{2}\ln 3 = b \qquad \text{Solve for } b.$$

Using $b = \frac{1}{2}\ln 3$ and the equation you found for $a$, you can determine that

$$a = \frac{100}{e^{2[(1/2)\ln 3]}} \qquad \text{Substitute } \tfrac{1}{2}\ln 3 \text{ for } b.$$

$$= \frac{100}{e^{\ln 3}} \qquad \text{Simplify.}$$

$$= \frac{100}{3} \qquad \text{Inverse Property}$$

$$\approx 33.33. \qquad \text{Simplify.}$$

So, with $a \approx 33.33$ and $b = \frac{1}{2}\ln 3 \approx 0.5493$, the exponential growth model is

$$y = 33.33e^{0.5493t}$$

as shown in Figure 3.32. This implies that, after 5 days, the population will be

$$y = 33.33e^{0.5493(5)} \approx 520 \text{ flies.}$$

CHECKPOINT Now try Exercise 37.

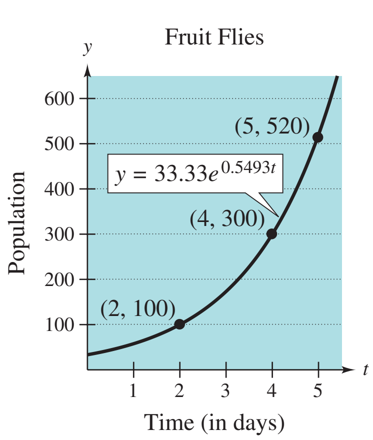


FIGURE 3.32

**Additional Example**

Radioactive iodine is a by-product of some types of nuclear reactors. Its half-life is about 60 days. That is, after 60 days, a given amount of radioactive iodine will have decayed to half the original amount. Suppose a contained nuclear accident occurs and gives off an initial amount $C$ of radioactive iodine.

a. Write an equation for the amount of radioactive iodine present at any time $t$ following the accident.

b. How long will it take for the radioactive iodine to decay to a level of 20% of the original amount?

*Solution*

a. Knowing that half the original amount remains after 60 days, you can use the exponential decay model $y = ae^{-bt}$ to obtain

$$\frac{1}{2}C = Ce^{-b(60)}$$

$$\frac{1}{2} = e^{-60b}$$

$$-\ln 2 = -60b$$

$$b = \frac{\ln 2}{60} \approx 0.0116.$$

So, $y = Ce^{-0.0116t}$.

b. The time required for the radioactive iodine to decay to 20% of the original amount is

$$Ce^{-0.0116t} = (0.2)C$$

$$e^{-0.0116t} = 0.2$$

$$-0.0116t = \ln 0.2$$

$$t = \frac{\ln 0.2}{-0.0116} \approx 139 \text{ days.}$$

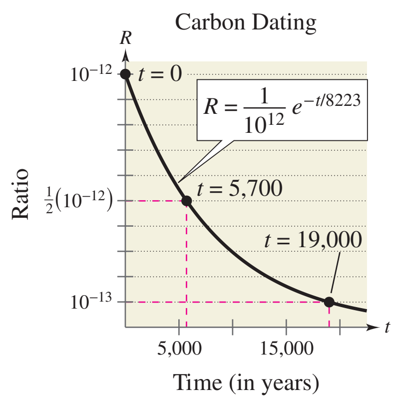



FIGURE 3.33

In living organic material, the ratio of the number of radioactive carbon isotopes (carbon 14) to the number of nonradioactive carbon isotopes (carbon 12) is about 1 to $10^{12}$. When organic material dies, its carbon 12 content remains fixed, whereas its radioactive carbon 14 begins to decay with a half-life of about 5700 years. To estimate the age of dead organic material, scientists use the following formula, which denotes the ratio of carbon 14 to carbon 12 present at any time $t$ (in years).

$$R = \frac{1}{10^{12}} e^{-t/8223} \qquad \text{Carbon dating model}$$

The graph of $R$ is shown in Figure 3.33. Note that $R$ decreases as $t$ increases.

## Example 3 Carbon Dating

Estimate the age of a newly discovered fossil in which the ratio of carbon 14 to carbon 12 is

$$R = \frac{1}{10^{13}}.$$

### Solution

In the carbon dating model, substitute the given value of $R$ to obtain the following.

$$\frac{1}{10^{12}} e^{-t/8223} = R \qquad \text{Write original model.}$$

$$\frac{e^{-t/8223}}{10^{12}} = \frac{1}{10^{13}} \qquad \text{Let } R = \frac{1}{10^{13}}.$$

$$e^{-t/8223} = \frac{1}{10} \qquad \text{Multiply each side by } 10^{12}.$$

$$\ln e^{-t/8223} = \ln \frac{1}{10} \qquad \text{Take natural log of each side.}$$

$$-\frac{t}{8223} \approx -2.3026 \qquad \text{Inverse Property}$$

$$t \approx 18{,}934 \qquad \text{Multiply each side by } -8223.$$

So, to the nearest thousand years, the age of the fossil is about 19,000 years.

CHECKPOINT Now try Exercise 41.

The value of $b$ in the exponential decay model $y = ae^{-bt}$ determines the *decay* of radioactive isotopes. For instance, to find how much of an initial 10 grams of $^{226}$Ra isotope with a half-life of 1599 years is left after 500 years, substitute this information into the model $y = ae^{-bt}$.

$$\frac{1}{2}(10) = 10e^{-b(1599)} \quad \Rightarrow \quad \ln \frac{1}{2} = -1599b \quad \Rightarrow \quad b = -\frac{\ln \frac{1}{2}}{1599}$$

Using the value of $b$ found above and $a = 10$, the amount left is

$$y = 10e^{-[-\ln(1/2)/1599](500)} \approx 8.05 \text{ grams.}$$

**STUDY TIP**

The carbon dating model in Example 3 assumed that the carbon 14 to carbon 12 ratio was one part in 10,000,000,000,000. Suppose an error in measurement occurred and the actual ratio was one part in 8,000,000,000,000. The fossil age corresponding to the actual ratio would then be approximately 17,000 years. Try checking this result.

## Gaussian Models

As mentioned at the beginning of this section, Gaussian models are of the form

$$y = ae^{-(x-b)^2/c}.$$

This type of model is commonly used in probability and statistics to represent populations that are **normally distributed.** The graph of a Gaussian model is called a **bell-shaped curve.** Try graphing the normal distribution with a graphing utility. Can you see why it is called a bell-shaped curve?

For *standard* normal distributions, the model takes the form

$$y = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}.$$

The **average value** for a population can be found from the bell-shaped curve by observing where the maximum $y$-value of the function occurs. The $x$-value corresponding to the maximum $y$-value of the function represents the average value of the independent variable—in this case, $x$.

### Example 4 SAT Scores

In 2004, the Scholastic Aptitude Test (SAT) math scores for college-bound seniors roughly followed the normal distribution given by

$$y = 0.0035e^{-(x-518)^2/25{,}992}, \quad 200 \le x \le 800$$

where $x$ is the SAT score for mathematics. Sketch the graph of this function. From the graph, estimate the average SAT score. (Source: College Board)

### Solution

The graph of the function is shown in Figure 3.34. On this bell-shaped curve, the maximum value of the curve represents the average score. From the graph, you can estimate that the average mathematics score for college-bound seniors in 2004 was 518.

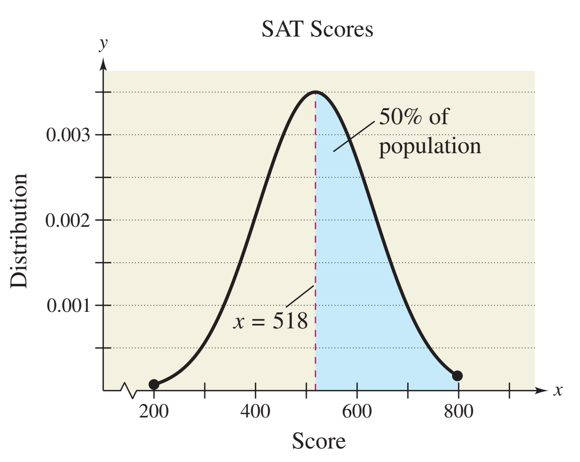



FIGURE 3.34

CHECKPOINT Now try Exercise 47.

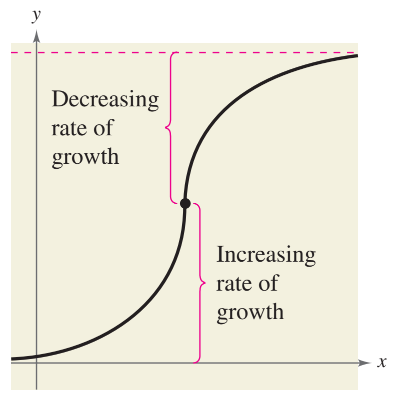



FIGURE 3.35

## Logistic Growth Models

Some populations initially have rapid growth, followed by a declining rate of growth, as indicated by the graph in Figure 3.35. One model for describing this type of growth pattern is the **logistic curve** given by the function

$$y = \frac{a}{1 + be^{-rx}}$$

where $y$ is the population size and $x$ is the time. An example is a bacteria culture that is initially allowed to grow under ideal conditions, and then under less favorable conditions that inhibit growth. A logistic growth curve is also called a **sigmoidal curve.**

### Example 5 Spread of a Virus

On a college campus of 5000 students, one student returns from vacation with a contagious and long-lasting flu virus. The spread of the virus is modeled by

$$y = \frac{5000}{1 + 4999e^{-0.8t}}, \quad t \geq 0$$

where $y$ is the total number of students infected after $t$ days. The college will cancel classes when 40% or more of the students are infected.

**a.** How many students are infected after 5 days?

**b.** After how many days will the college cancel classes?

### Solution

**a.** After 5 days, the number of students infected is

$$y = \frac{5000}{1 + 4999e^{-0.8(5)}} = \frac{5000}{1 + 4999e^{-4}} \approx 54.$$

**b.** Classes are canceled when the number infected is $(0.40)(5000) = 2000$.

$$2000 = \frac{5000}{1 + 4999e^{-0.8t}}$$

$$1 + 4999e^{-0.8t} = 2.5$$

$$e^{-0.8t} = \frac{1.5}{4999}$$

$$\ln e^{-0.8t} = \ln \frac{1.5}{4999}$$

$$-0.8t = \ln \frac{1.5}{4999}$$

$$t = -\frac{1}{0.8} \ln \frac{1.5}{4999}$$

$$t \approx 10.1$$

So, after about 10 days, at least 40% of the students will be infected, and the college will cancel classes. The graph of the function is shown in Figure 3.36.

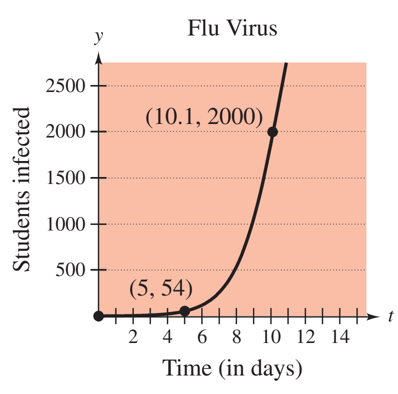



FIGURE 3.36

CHECKPOINT Now try Exercise 49.

## Logarithmic Models

### Example 6 Magnitudes of Earthquakes

On the Richter scale, the magnitude $R$ of an earthquake of intensity $I$ is given by

$$R = \log \frac{I}{I_0}$$

where $I_0 = 1$ is the minimum intensity used for comparison. Find the intensities per unit of area for each earthquake. (Intensity is a measure of the wave energy of an earthquake.)

**a.** Northern Sumatra in 2004: $R = 9.0$

**b.** Southeastern Alaska in 2004: $R = 6.8$

### Solution

**a.** Because $I_0 = 1$ and $R = 9.0$, you have

$$9.0 = \log \frac{I}{1} \qquad \text{Substitute 1 for } I_0 \text{ and 9.0 for } R.$$

$$10^{9.0} = 10^{\log I} \qquad \text{Exponentiate each side.}$$

$$I = 10^{9.0} \approx 100{,}000{,}000. \qquad \text{Inverse Property}$$

**b.** For $R = 6.8$, you have

$$6.8 = \log \frac{I}{1} \qquad \text{Substitute 1 for } I_0 \text{ and 6.8 for } R.$$

$$10^{6.8} = 10^{\log I} \qquad \text{Exponentiate each side.}$$

$$I = 10^{6.8} \approx 6{,}310{,}000. \qquad \text{Inverse Property}$$

Note that an increase of 2.2 units on the Richter scale (from 6.8 to 9.0) represents an increase in intensity by a factor of

$$\frac{1{,}000{,}000{,}000}{6{,}310{,}000} \approx 158.$$

In other words, the intensity of the earthquake in Sumatra was about 158 times greater than that of the earthquake in Alaska.

CHECKPOINT Now try Exercise 51.

BAY ISMOYO/AFP/Getty Images

*On December 26, 2004, an earthquake of magnitude 9.0 struck northern Sumatra and many other Asian countries. This earthquake caused a deadly tsunami and was the fourth largest earthquake in the world since 1900.*

**Alternative Writing About Mathematics**

Use your school's library, the Internet, or some other reference source to find an application that fits one of the five models discussed in this section. After you have collected data for the model, plot the corresponding points and find an equation that describes the points you have plotted.

| $t$ | Year | Population, $P$ |
|---|---|---|
| 1 | 1910 | 92.23 |
| 2 | 1920 | 106.02 |
| 3 | 1930 | 123.20 |
| 4 | 1940 | 132.16 |
| 5 | 1950 | 151.33 |
| 6 | 1960 | 179.32 |
| 7 | 1970 | 203.30 |
| 8 | 1980 | 226.54 |
| 9 | 1990 | 248.72 |
| 10 | 2000 | 281.42 |

### Writing About Mathematics

**Comparing Population Models** The populations $P$ (in millions) of the United States for the census years from 1910 to 2000 are shown in the table at the left. Least squares regression analysis gives the best quadratic model for these data as $P = 1.0328t^2 + 9.607t + 81.82$, and the best exponential model for these data as $P = 82.677e^{0.124t}$. Which model better fits the data? Describe how you reached your conclusion. (Source: U.S. Census Bureau)

# 3.5 Exercises

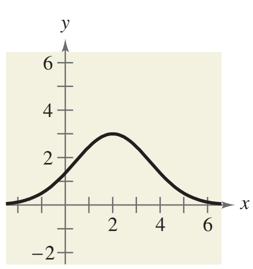
**VOCABULARY CHECK:** Fill in the blanks.

1. An exponential growth model has the form ________ and an exponential decay model has the form ________.
2. A logarithmic model has the form ________ or ________.
3. Gaussian models are commonly used in probability and statistics to represent populations that are ________ ________.
4. The graph of a Gaussian model is ________ shaped, where the ________ ________ is the maximum $y$-value of the graph.
5. A logistic curve is also called a ________ curve.

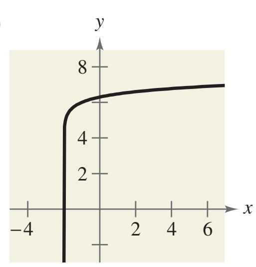
**PREREQUISITE SKILLS REVIEW:** Practice and review algebra skills needed for this section at **www.Eduspace.com.**

**In Exercises 1–6, match the function with its graph. [The graphs are labeled (a), (b), (c), (d), (e), and (f).]**

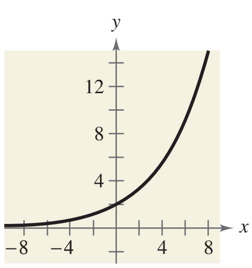
(a)



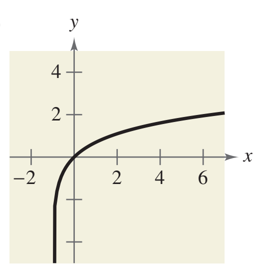

(b)



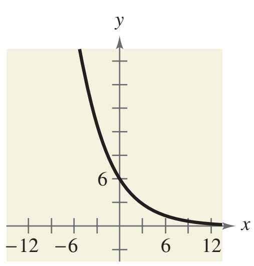

(c)



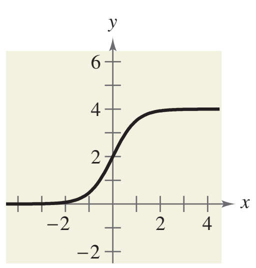

(d)




(e)




(f)




**1.** $y = 2e^{x/4}$

**2.** $y = 6e^{-x/4}$

**3.** $y = 6 + \log(x + 2)$

**4.** $y = 3e^{-(x-2)^2/5}$

**5.** $y = \ln(x + 1)$

**6.** $y = \dfrac{4}{1 + e^{-2x}}$

***Compound Interest*** **In Exercises 7–14, complete the table for a savings account in which interest is compounded continuously.**

| | *Initial Investment* | *Annual % Rate* | *Time to Double* | *Amount After 10 Years* |
|---|---|---|---|---|
| **7.** | \$1000 | 3.5% | | |
| **8.** | \$750 | $10\frac{1}{2}\%$ | | |
| **9.** | \$750 | | $7\frac{3}{4}$ yr | |
| **10.** | \$10,000 | | 12 yr | |
| **11.** | \$500 | | | \$1505.00 |
| **12.** | \$600 | | | \$19,205.00 |
| **13.** | | 4.5% | | \$10,000.00 |
| **14.** | | 2% | | \$2000.00 |

***Compound Interest*** **In Exercises 15 and 16, determine the principal $P$ that must be invested at rate $r$, compounded monthly, so that \$500,000 will be available for retirement in $t$ years.**

**15.** $r = 7\frac{1}{2}\%, t = 20$

**16.** $r = 12\%, t = 40$

***Compound Interest*** **In Exercises 17 and 18, determine the time necessary for \$1000 to double if it is invested at interest rate $r$ compounded (a) annually, (b) monthly, (c) daily, and (d) continuously.**

**17.** $r = 11\%$

**18.** $r = 10\frac{1}{2}\%$

**19.** ***Compound Interest*** Complete the table for the time $t$ necessary for $P$ dollars to triple if interest is compounded continuously at rate $r$.

| $r$ | 2% | 4% | 6% | 8% | 10% | 12% |
|---|---|---|---|---|---|---|
| $t$ | | | | | | |

**20.** ***Modeling Data*** Draw a scatter plot of the data in Exercise 19. Use the *regression* feature of a graphing utility to find a model for the data.

**21.** ***Compound Interest*** Complete the table for the time $t$ necessary for $P$ dollars to triple if interest is compounded annually at rate $r$.

| $r$ | 2% | 4% | 6% | 8% | 10% | 12% |
|---|---|---|---|---|---|---|
| $t$ | | | | | | |

**22.** ***Modeling Data*** Draw a scatter plot of the data in Exercise 21. Use the *regression* feature of a graphing utility to find a model for the data.

**23.** ***Comparing Models*** If \$1 is invested in an account over a 10-year period, the amount in the account, where $t$ represents the time in years, is given by $A = 1 + 0.075[\![t]\!]$ or $A = e^{0.07t}$ depending on whether the account pays simple interest at $7\frac{1}{2}\%$ or continuous compound interest at 7%. Graph each function on the same set of axes. Which grows at a higher rate? (Remember that $[\![t]\!]$ is the greatest integer function discussed in Section 1.6.)

**24.** ***Comparing Models*** If \$1 is invested in an account over a 10-year period, the amount in the account, where $t$ represents the time in years, is given by

$$A = 1 + 0.06[\![t]\!] \quad \text{or} \quad A = \left(1 + \frac{0.055}{365}\right)^{[\![365t]\!]}$$

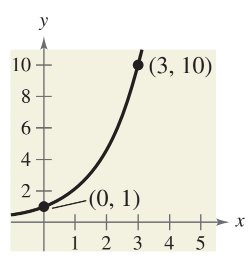
depending on whether the account pays simple interest at 6% or compound interest at $5\frac{1}{2}\%$ compounded daily. Use a graphing utility to graph each function in the same viewing window. Which grows at a higher rate?

***Radioactive Decay*** **In Exercises 25–30, complete the table for the radioactive isotope.**

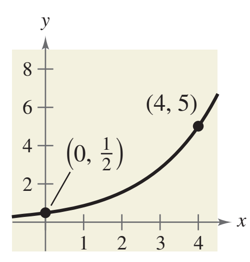
| | *Isotope* | *Half-life (years)* | *Initial Quantity* | *Amount After 1000 Years* |
|---|---|---|---|---|
| **25.** | $^{226}Ra$ | 1599 | 10 g | |
| **26.** | $^{226}Ra$ | 1599 | | 1.5 g |
| **27.** | $^{14}C$ | 5715 | | 2 g |
| **28.** | $^{14}C$ | 5715 | 3 g | |
| **29.** | $^{239}Pu$ | 24,100 | | 2.1 g |
| **30.** | $^{239}Pu$ | 24,100 | | 0.4 g |

**In Exercises 31–34, find the exponential model $y = ae^{bx}$ that fits the points shown in the graph or table.**

**31.**




**32.**




**33.**

| $x$ | 0 | 4 |
|---|---|---|
| $y$ | 5 | 1 |

**34.**

| $x$ | 0 | 3 |
|---|---|---|
| $y$ | 1 | $\frac{1}{4}$ |

**35.** ***Population*** The population $P$ (in thousands) of Pittsburgh, Pennsylvania from 2000 through 2003 can be modeled by $P = 2430e^{-0.0029t}$, where $t$ represents the year, with $t = 0$ corresponding to 2000. (Source: U.S. Census Bureau)

(a) According to the model, was the population of Pittsburgh increasing or decreasing from 2000 to 2003? Explain your reasoning.

(b) What were the populations of Pittsburgh in 2000 and 2003?

(c) According to the model, when will the population be approximately 2.3 million?

## Model It

**36.** ***Population*** The table shows the populations (in millions) of five countries in 2000 and the projected populations (in millions) for the year 2010. (Source: U.S. Census Bureau)

| Country | 2000 | 2010 |
|---|---|---|
| Bulgaria | 7.8 | 7.1 |
| Canada | 31.3 | 34.3 |
| China | 1268.9 | 1347.6 |
| United Kingdom | 59.5 | 61.2 |
| United States | 282.3 | 309.2 |

(a) Find the exponential growth or decay model $y = ae^{bt}$ or $y = ae^{-bt}$ for the population of each country by letting $t = 0$ correspond to 2000. Use the model to predict the population of each country in 2030.

(b) You can see that the populations of the United States and the United Kingdom are growing at different rates. What constant in the equation $y = ae^{bt}$ is determined by these different growth rates? Discuss the relationship between the different growth rates and the magnitude of the constant.

(c) You can see that the population of China is increasing while the population of Bulgaria is decreasing. What constant in the equation $y = ae^{bt}$ reflects this difference? Explain.

**37.** ***Website Growth*** The number $y$ of hits a new search-engine website receives each month can be modeled by

$$y = 4080e^{kt}$$

where $t$ represents the number of months the website has been operating. In the website's third month, there were 10,000 hits. Find the value of $k$, and use this result to predict the number of hits the website will receive after 24 months.

**38.** ***Value of a Painting*** The value $V$ (in millions of dollars) of a famous painting can be modeled by

$$V = 10e^{kt}$$

where $t$ represents the year, with $t = 0$ corresponding to 1990. In 2004, the same painting was sold for \$65 million. Find the value of $k$, and use this result to predict the value of the painting in 2010.

**39.** ***Bacteria Growth*** The number $N$ of bacteria in a culture is modeled by

$$N = 100e^{kt}$$

where $t$ is the time in hours. If $N = 300$ when $t = 5$, estimate the time required for the population to double in size.

**40.** ***Bacteria Growth*** The number $N$ of bacteria in a culture is modeled by

$$N = 250e^{kt}$$

where $t$ is the time in hours. If $N = 280$ when $t = 10$, estimate the time required for the population to double in size.

**41.** ***Carbon Dating***

(a) The ratio of carbon 14 to carbon 12 in a piece of wood discovered in a cave is $R = 1/8^{14}$. Estimate the age of the piece of wood.

(b) The ratio of carbon 14 to carbon 12 in a piece of paper buried in a tomb is $R = 1/13^{11}$. Estimate the age of the piece of paper.

**42.** ***Radioactive Decay*** Carbon 14 dating assumes that the carbon dioxide on Earth today has the same radioactive content as it did centuries ago. If this is true, the amount of $^{14}C$ absorbed by a tree that grew several centuries ago should be the same as the amount of $^{14}C$ absorbed by a tree growing today. A piece of ancient charcoal contains only 15% as much radioactive carbon as a piece of modern charcoal. How long ago was the tree burned to make the ancient charcoal if the half-life of $^{14}C$ is 5715 years?

**43.** ***Depreciation*** A 2005 Jeep Wrangler that costs \$30,788 new has a book value of \$18,000 after 2 years.

(a) Find the linear model $V = mt + b$.

(b) Find the exponential model $V = ae^{kt}$.

(c) Use a graphing utility to graph the two models in the same viewing window. Which model depreciates faster in the first 2 years?

(d) Find the book values of the vehicle after 1 year and after 3 years using each model.

(e) Explain the advantages and disadvantages of using each model to a buyer and a seller.

**44.** ***Depreciation*** A Dell Inspiron 8600 laptop computer that costs \$1150 new has a book value of \$550 after 2 years.

(a) Find the linear model $V = mt + b$.

(b) Find the exponential model $V = ae^{kt}$.

(c) Use a graphing utility to graph the two models in the same viewing window. Which model depreciates faster in the first 2 years?

(d) Find the book values of the computer after 1 year and after 3 years using each model.

(e) Explain the advantages and disadvantages to a buyer and a seller of using each model.

**45.** ***Sales*** The sales $S$ (in thousands of units) of a new CD burner after it has been on the market for $t$ years are modeled by

$$S(t) = 100(1 - e^{kt}).$$

Fifteen thousand units of the new product were sold the first year.

(a) Complete the model by solving for $k$.

(b) Sketch the graph of the model.

(c) Use the model to estimate the number of units sold after 5 years.

**46.** ***Learning Curve*** The management at a plastics factory has found that the maximum number of units a worker can produce in a day is 30. The learning curve for the number $N$ of units produced per day after a new employee has worked $t$ days is modeled by

$$N = 30(1 - e^{kt}).$$

After 20 days on the job, a new employee produces 19 units.

(a) Find the learning curve for this employee (first, find the value of $k$).

(b) How many days should pass before this employee is producing 25 units per day?

**47.** ***IQ Scores*** The IQ scores from a sample of a class of returning adult students at a small northeastern college roughly follow the normal distribution

$$y = 0.0266e^{-(x-100)^2/450}, \quad 70 \le x \le 115$$

where $x$ is the IQ score.

(a) Use a graphing utility to graph the function.

(b) From the graph in part (a), estimate the average IQ score of an adult student.

**48.** ***Education*** The time (in hours per week) a student utilizes a math-tutoring center roughly follows the normal distribution

$$y = 0.7979e^{-(x-5.4)^2/0.5}, \quad 4 \le x \le 7$$

where $x$ is the number of hours.

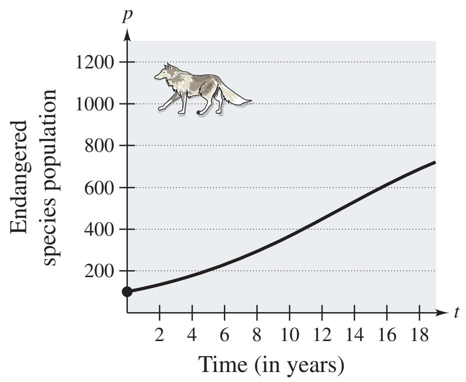
(a) Use a graphing utility to graph the function.

(b) From the graph in part (a), estimate the average number of hours per week a student uses the tutor center.

**49.** ***Population Growth*** A conservation organization releases 100 animals of an endangered species into a game preserve. The organization believes that the preserve has a carrying capacity of 1000 animals and that the growth of the pack will be modeled by the logistic curve

$$p(t) = \frac{1000}{1 + 9e^{-0.1656t}}$$

where $t$ is measured in months (see figure).




(a) Estimate the population after 5 months.

(b) After how many months will the population be 500?

(c) Use a graphing utility to graph the function. Use the graph to determine the horizontal asymptotes, and interpret the meaning of the larger $p$-value in the context of the problem.

**50.** ***Sales*** After discontinuing all advertising for a tool kit in 2000, the manufacturer noted that sales began to drop according to the model

$$S = \frac{500{,}000}{1 + 0.6e^{kt}}$$

where $S$ represents the number of units sold and $t = 0$ represents 2000. In 2004, the company sold 300,000 units.

(a) Complete the model by solving for $k$.

(b) Estimate sales in 2008.

***Geology*** **In Exercises 51 and 52, use the Richter scale**

$$R = \log \frac{I}{I_0}$$

**for measuring the magnitudes of earthquakes.**

**51.** Find the intensity $I$ of an earthquake measuring $R$ on the Richter scale (let $I_0 = 1$).

(a) Centeral Alaska in 2002, $R = 7.9$

(b) Hokkaido, Japan in 2003, $R = 8.3$

(c) Illinois in 2004, $R = 4.2$

**52.** Find the magnitude $R$ of each earthquake of intensity $I$ (let $I_0 = 1$).

(a) $I = 80{,}500{,}000$ (b) $I = 48{,}275{,}000$

(c) $I = 251{,}200$

***Intensity of Sound*** **In Exercises 53–56, use the following information for determining sound intensity. The level of sound $\beta$, in decibels, with an intensity of $I$, is given by**

$$\beta = 10 \log \frac{I}{I_0}$$

**where $I_0$ is an intensity of $10^{-12}$ watt per square meter, corresponding roughly to the faintest sound that can be heard by the human ear. In Exercises 53 and 54, find the level of sound $\beta$.**

**53.** (a) $I = 10^{-10}$ watt per m$^2$ (quiet room)

(b) $I = 10^{-5}$ watt per m$^2$ (busy street corner)

(c) $I = 10^{-8}$ watt per m$^2$ (quiet radio)

(d) $I = 10^{0}$ watt per m$^2$ (threshold of pain)

**54.** (a) $I = 10^{-11}$ watt per m$^2$ (rustle of leaves)

(b) $I = 10^{2}$ watt per m$^2$ (jet at 30 meters)

(c) $I = 10^{-4}$ watt per m$^2$ (door slamming)

(d) $I = 10^{-2}$ watt per m$^2$ (siren at 30 meters)

**55.** Due to the installation of noise suppression materials, the noise level in an auditorium was reduced from 93 to 80 decibels. Find the percent decrease in the intensity level of the noise as a result of the installation of these materials.

**56.** Due to the installation of a muffler, the noise level of an engine was reduced from 88 to 72 decibels. Find the percent decrease in the intensity level of the noise as a result of the installation of the muffler.

***pH Levels*** **In Exercises 57–62, use the acidity model given by pH $= -\log[H^+]$, where acidity (pH) is a measure of the hydrogen ion concentration $[H^+]$ (measured in moles of hydrogen per liter) of a solution.**

**57.** Find the pH if $[H^+] = 2.3 \times 10^{-5}$.

**58.** Find the pH if $[H^+] = 11.3 \times 10^{-6}$.

**59.** Compute $[H^+]$ for a solution in which $\text{pH} = 5.8$.

**60.** Compute $[H^+]$ for a solution in which $\text{pH} = 3.2$.

**61.** Apple juice has a pH of 2.9 and drinking water has a pH of 8.0. The hydrogen ion concentration of the apple juice is how many times the concentration of drinking water?

**62.** The pH of a solution is decreased by one unit. The hydrogen ion concentration is increased by what factor?

**63.** ***Forensics*** At 8:30 A.M., a coroner was called to the home of a person who had died during the night. In order to estimate the time of death, the coroner took the person's temperature twice. At 9:00 A.M. the temperature was 85.7°F, and at 11:00 a.m. the temperature was 82.8°F. From these two temperatures, the coroner was able to determine that the time elapsed since death and the body temperature were related by the formula

$$t = -10 \ln \frac{T - 70}{98.6 - 70}$$

where $t$ is the time in hours elapsed since the person died and $T$ is the temperature (in degrees Fahrenheit) of the person's body. Assume that the person had a normal body temperature of 98.6°F at death, and that the room temperature was a constant 70°F. (This formula is derived from a general cooling principle called *Newton's Law of Cooling*.) Use the formula to estimate the time of death of the person.

**64.** ***Home Mortgage*** A \$120,000 home mortgage for 35 years at $7\frac{1}{2}\%$ has a monthly payment of \$809.39. Part of the monthly payment is paid toward the interest charge on the unpaid balance, and the remainder of the payment is used to reduce the principal. The amount that is paid toward the interest is

$$u = M - \left(M - \frac{Pr}{12}\right)\left(1 + \frac{r}{12}\right)^{12t}$$

and the amount that is paid toward the reduction of the principal is

$$v = \left(M - \frac{Pr}{12}\right)\left(1 + \frac{r}{12}\right)^{12t}.$$

In these formulas, $P$ is the size of the mortgage, $r$ is the interest rate, $M$ is the monthly payment, and $t$ is the time in years.

(a) Use a graphing utility to graph each function in the same viewing window. (The viewing window should show all 35 years of mortgage payments.)

(b) In the early years of the mortgage, is the larger part of the monthly payment paid toward the interest or the principal? Approximate the time when the monthly payment is evenly divided between interest and principal reduction.

(c) Repeat parts (a) and (b) for a repayment period of 20 years ($M = \$966.71$). What can you conclude?

**65.** ***Home Mortgage*** The total interest $u$ paid on a home mortgage of $P$ dollars at interest rate $r$ for $t$ years is

$$u = P\left[\frac{rt}{1 - \left(\dfrac{1}{1 + r/12}\right)^{12t}} - 1\right].$$

Consider a \$120,000 home mortgage at $7\frac{1}{2}\%$.

(a) Use a graphing utility to graph the total interest function.

(b) Approximate the length of the mortgage for which the total interest paid is the same as the size of the mortgage. Is it possible that some people are paying twice as much in interest charges as the size of the mortgage?

**66.** ***Data Analysis*** The table shows the time $t$ (in seconds) required to attain a speed of $s$ miles per hour from a standing start for a car.

| Speed, $s$ | Time, $t$ |
|---|---|
| 30 | 3.4 |
| 40 | 5.0 |
| 50 | 7.0 |
| 60 | 9.3 |
| 70 | 12.0 |
| 80 | 15.8 |
| 90 | 20.0 |

Two models for these data are as follows.

$$t_1 = 40.757 + 0.556s - 15.817 \ln s$$

$$t_2 = 1.2259 + 0.0023s^2$$

(a) Use the *regression* feature of a graphing utility to find a linear model $t_3$ and an exponential model $t_4$ for the data.

(b) Use a graphing utility to graph the data and each model in the same viewing window.

(c) Create a table comparing the data with estimates obtained from each model.

(d) Use the results of part (c) to find the sum of the absolute values of the differences between the data and the estimated values given by each model. Based on the four sums, which model do you think better fits the data? Explain.

## Synthesis

***True or False?*** **In Exercises 67–70, determine whether the statement is true or false. Justify your answer.**

**67.** The domain of a logistic growth function cannot be the set of real numbers.

**68.** A logistic growth function will always have an $x$-intercept.

**69.** The graph of

$$f(x) = \frac{4}{1 + 6e^{-2x}} + 5$$

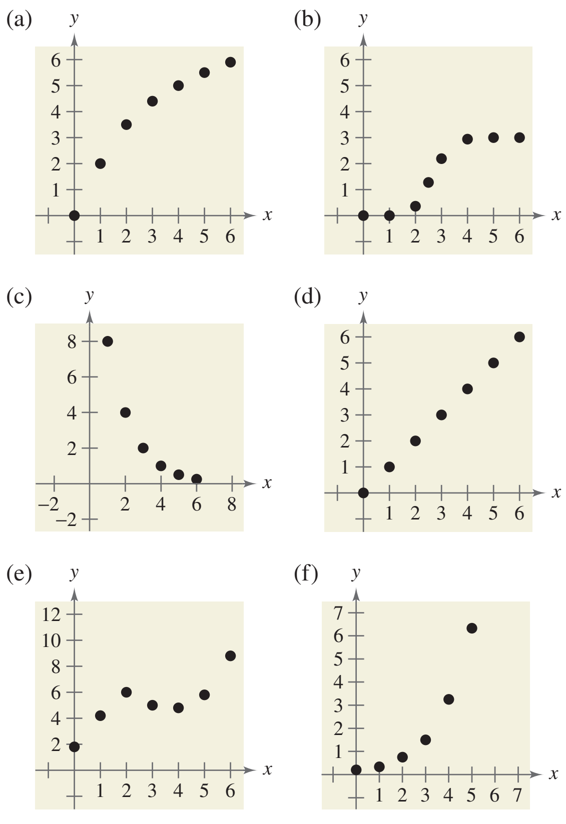
is the graph of

$$g(x) = \frac{4}{1 + 6e^{-2x}}$$

shifted to the right five units.

**70.** The graph of a Gaussian model will never have an $x$-intercept.

**71.** Identify each model as linear, logarithmic, exponential, logistic, or none of the above. Explain your reasoning.




**72.** ***Writing*** Use your school's library, the Internet, or some other reference source to write a paper describing John Napier's work with logarithms.

## Skills Review

**In Exercises 73–78, (a) plot the points, (b) find the distance between the points, (c) find the midpoint of the line segment joining the points, and (d) find the slope of the line passing through the points.**

**73.** $(-1, 2), (0, 5)$

**74.** $(4, -3), (-6, 1)$

**75.** $(3, 3), (14, -2)$

**76.** $(7, 0), (10, 4)$

**77.** $\left(\frac{1}{2}, -\frac{1}{4}\right), \left(\frac{3}{4}, 0\right)$

**78.** $\left(\frac{7}{3}, \frac{1}{6}\right), \left(-\frac{2}{3}, -\frac{1}{3}\right)$

**In Exercises 79–88, sketch the graph of the equation.**

**79.** $y = 10 - 3x$

**80.** $y = -4x - 1$

**81.** $y = -2x^2 - 3$

**82.** $y = 2x^2 - 7x - 30$

**83.** $3x^2 - 4y = 0$

**84.** $-x^2 - 8y = 0$

**85.** $y = \dfrac{4}{1 - 3x}$

**86.** $y = \dfrac{x^2}{-x - 2}$

**87.** $x^2 + (y - 8)^2 = 25$

**88.** $(x - 4)^2 + (y + 7) = 4$

**In Exercises 89–92, graph the exponential function.**

**89.** $f(x) = 2^{x-1} + 5$

**90.** $f(x) = -2^{-x-1} - 1$

**91.** $f(x) = 3^x - 4$

**92.** $f(x) = -3^x + 4$

**93.** **Make a Decision** To work an extended application analyzing the net sales for Kohl's Corporation from 1992 to 2004, visit this text's website at *college.hmco.com*. *(Data Source: Kohl's Illinois, Inc.)*