

Verifying Trigonometric Identities

Objective: To verify that two expressions are equivalent. That is, we want to verify that what we have is an identity.

- To do this, we generally pick the expression on one side of the given identity and manipulate that expression until we get the other side.
- In most cases, it is best to start with the more complex looking side and try to simply to match the less complex side.
- You must be very familiar with the fundamental trigonometric identities, especially the Pythagorean Identities. In some cases, a direct substitution using these fundamental identities will verify the identity you are trying to prove (Exercise 8 at the end of this document is one example).
- Some special approaches are useful for certain types of identities, which are provided below.

Identity Type	Verification	Approach
<p>Type 1:</p> <p>Sometimes it is easier if we just rewrite everything in terms of sine and cosine to see if the expression simplifies.</p>	<p>Verify:</p> $\cot x + 1 = \csc x (\cos x + \sin x)$ $\begin{aligned} \text{RHS} \rightarrow \csc x (\cos x + \sin x) &= \frac{1}{\sin x} (\cos x + \sin x) \\ &= \frac{\cos x}{\sin x} + \frac{\sin x}{\sin x} \\ &= \cot x + 1 \end{aligned}$	<ul style="list-style-type: none"> • Start with more complex RHS. • Rewrite $\csc x$ in terms of sine or cosine. • Remember, $\csc x = 1/\sin x$ • Also note, $\cos x/\sin x = \cot x$ • The RHS simplifies to original LHS.
<p>Type 2:</p> <p>In some cases, the more complex side involves a fraction that can be split up. Then we rewrite everything in terms of sine and cosine.</p>	<p>Verify:</p> $\frac{\tan t - \cot t}{\sin t \cos t} = \sec^2 t - \csc^2 t$ $\begin{aligned} \text{LHS} \rightarrow \frac{\tan t - \cot t}{\sin t \cos t} &= \frac{\tan t}{\sin t \cos t} - \frac{\cot t}{\sin t \cos t} \\ &= \tan t \cdot \frac{1}{\sin t \cos t} - \cot t \cdot \frac{1}{\sin t \cos t} \\ &= \frac{\sin t}{\cos t} \cdot \frac{1}{\sin t \cos t} - \frac{\cos t}{\sin t} \cdot \frac{1}{\sin t \cos t} \\ &= \frac{1}{\cos^2 t} - \frac{1}{\sin^2 t} \\ &= \sec^2 t - \csc^2 t \end{aligned}$	<ul style="list-style-type: none"> • Start with the more complex LHS. • Rewrite the LHS as difference of two fractions. • Split out $\tan t$ and $\cot t$ to make it easier to simplify. • Notice in the first term, the $\sin t$ cancels out; and in the second term, $\cos t$ cancels out. • The new terms are reciprocal identities • The LHS simplifies to the original RHS.

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<p><u>Type 3:</u></p> <p>Using the property of conjugates is sometimes helpful. For an expression like $a + b$, the conjugate would be $a - b$. When you multiply conjugates, you often get a more useful expression, e.g., $(a + b)(a - b)$. Sometimes multiplying by the conjugate will simplify an expression and help in verifying the given identity.</p>	<p>Verify:</p> $\frac{\cos x}{1 - \sin x} = \frac{1 + \sin x}{\cos x}$ $\text{RHS} \rightarrow \frac{1 + \sin x}{\cos x} = \frac{1 + \sin x}{\cos x} \left(\frac{1 - \sin x}{1 - \sin x} \right)$ $= \frac{1 - \sin^2 x}{\cos x (1 - \sin x)}$ $= \frac{\cos^2 x}{\cos x (1 - \sin x)}$ $= \frac{\cos x \cos x}{\cos x (1 - \sin x)}$ $= \frac{\cos x}{1 - \sin x}$	<ul style="list-style-type: none"> • We could start with either side; but here we will start with the RHS. • The conjugate of the numerator $1 + \sin x$ is $1 - \sin x$. • Multiply by $\frac{1 - \sin x}{1 - \sin x} = 1$ • Remember, $1 - \sin^2 x = \cos^2 x$ • Once we reduce the fraction, we get the LHS of the original identity.
<p><u>Type 4:</u></p> <p>Combining fractions before using identities may be an appropriate strategy.</p>	<p>Verify:</p> $\frac{1}{1 - \sin \alpha} + \frac{1}{1 + \sin \alpha} = 2 \sec^2 \alpha$ $\text{LHS} \rightarrow \frac{1}{1 - \sin \alpha} + \frac{1}{1 + \sin \alpha} = \frac{1}{1 - \sin \alpha} \left(\frac{1 + \sin \alpha}{1 + \sin \alpha} \right) + \frac{1}{1 + \sin \alpha} \left(\frac{1 - \sin \alpha}{1 - \sin \alpha} \right)$ $= \frac{(1 + \sin \alpha) + (1 - \sin \alpha)}{(1 - \sin \alpha)(1 + \sin \alpha)}$ $= \frac{2}{1 - \sin^2 \alpha}$ $= \frac{2}{\cos^2 \alpha}$ $= 2 \sec^2 \alpha$	<ul style="list-style-type: none"> • Notice that the denominators of the fractions on the LHS are conjugates. • So we will use the property of conjugates to combine the LHS fractions and simplify.

Verifying Trigonometric Identities

Verify the following trigonometric identities.

1. $\cos x + \sin x \tan x = \sec x$

2. $\frac{\csc x - \sin x}{\sin x \csc x} = \csc x - \sin x$

3. $\frac{1}{\tan \beta} + \tan \beta = \frac{\sec^2 \beta}{\tan \beta}$

4. $\frac{1 + \sin \theta}{\cos \theta} + \frac{\cos \theta}{1 + \sin \theta} = 2 \sec \theta$

5. $\sec y + \tan y = \frac{\cos y}{1 - \sin y}$

6. $\frac{\cos^2 x - \sin^2 x}{1 - \tan^2 x} = \cos^2 x$

7. $\frac{\sin x}{\cos x + 1} + \frac{\cos x - 1}{\sin x} = 0$

8. $\frac{\sin^2 \theta + \cos^2 \theta + \cot^2 \theta}{1 + \tan^2 \theta} = \cot^2 \theta$

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Solutions to Exercises

$$\begin{aligned} 1. \text{ LHS} \rightarrow \cos x + \sin x \tan x &= \cos x + \sin x \left(\frac{\sin x}{\cos x} \right) \\ &= \cos x + \frac{\sin^2 x}{\cos x} \\ &= \frac{\cos^2 x}{\cos x} + \frac{\sin^2 x}{\cos x} \\ &= \frac{\cos^2 x + \sin^2 x}{\cos x} \\ &= \frac{1}{\cos x} \\ &= \sec x \end{aligned}$$

$$\begin{aligned} 3. \text{ LHS} \rightarrow \frac{1}{\tan \beta} + \tan \beta &= \frac{1}{\tan \beta} + \frac{\tan^2 \beta}{\tan \beta} \\ &= \frac{1 + \tan^2 \beta}{\tan \beta} \\ &= \frac{\sec^2 \beta}{\tan \beta} \end{aligned}$$

$$\begin{aligned} 2. \text{ LHS} \rightarrow \frac{\csc x - \sin x}{\sin x \csc x} &= \frac{1}{\sin x \csc x} (\csc x - \sin x) \\ &= \frac{1}{\sin x \csc x} \csc x - \frac{1}{\sin x \csc x} \sin x \\ &= \frac{1}{\sin x} - \frac{1}{\csc x} \\ &= \csc x - \sin x \end{aligned}$$

$$\begin{aligned} 4. \text{ LHS} \rightarrow \frac{1 + \sin \theta}{\cos \theta} + \frac{\cos \theta}{1 + \sin \theta} &= \frac{1 + \sin \theta}{\cos \theta} \left(\frac{1 + \sin \theta}{1 + \sin \theta} \right) + \frac{\cos \theta}{1 + \sin \theta} \left(\frac{\cos \theta}{\cos \theta} \right) \\ &= \frac{(1 + \sin \theta)^2 + \cos^2 \theta}{\cos \theta (1 + \sin \theta)} \\ &= \frac{1 + 2\sin \theta + \sin^2 \theta + \cos^2 \theta}{\cos \theta (1 + \sin \theta)} \\ &= \frac{1 + 2\sin \theta + 1}{\cos \theta (1 + \sin \theta)} \\ &= \frac{2 + 2\sin \theta}{\cos \theta (1 + \sin \theta)} \\ &= \frac{2(1 + \sin \theta)}{\cos \theta (1 + \sin \theta)} \\ &= \frac{2}{\cos \theta} \\ &= 2\sec \theta \end{aligned}$$

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$$\begin{aligned}
 5. \text{ RHS} &\rightarrow \frac{\cos y}{1 - \sin y} = \frac{\cos y}{1 - \sin y} \left(\frac{1 + \sin y}{1 + \sin y} \right) \\
 &= \frac{\cos y(1 + \sin y)}{1 - \sin^2 y} \\
 &= \frac{\cos y(1 + \sin y)}{\cos^2 y} \\
 &= \frac{1 + \sin y}{\cos y} \\
 &= \frac{1}{\cos y} + \frac{\sin y}{\cos y} \\
 &= \sec y + \tan y
 \end{aligned}$$

$$\begin{aligned}
 7. \text{ LHS} &\rightarrow \frac{\sin x}{\cos x + 1} + \frac{\cos x - 1}{\sin x} = \frac{\sin x}{\cos x + 1} \left(\frac{\sin x}{\sin x} \right) + \frac{\cos x - 1}{\sin x} \left(\frac{\cos x + 1}{\cos x + 1} \right) \\
 &= \frac{\sin^2 x}{\sin x(\cos x + 1)} + \frac{\cos^2 x - 1}{\sin x(\cos x + 1)} \\
 &= \frac{\sin^2 x + \cos^2 x - 1}{\sin x(\cos x + 1)} \\
 &= \frac{1 - 1}{\sin x(\cos x + 1)} \\
 &= \frac{0}{\sin x(\cos x + 1)} \\
 &= 0
 \end{aligned}$$

$$\begin{aligned}
 6. \text{ LHS} &\rightarrow \frac{\cos^2 x - \sin^2 x}{1 - \tan^2 x} = \frac{\cos^2 x - \sin^2 x}{1 - \frac{\sin^2 x}{\cos^2 x}} \\
 &= \frac{\cos^2 x - \sin^2 x}{\frac{\cos^2 x - \sin^2 x}{\cos^2 x}} \\
 &= (\cos^2 x - \sin^2 x) \left(\frac{\cos^2 x}{\cos^2 x - \sin^2 x} \right) \\
 &= \cos^2 x
 \end{aligned}$$

$$\begin{aligned}
 8. \text{ LHS} &\rightarrow \frac{\sin^2 \theta + \cos^2 \theta + \cot^2 \theta}{1 + \tan^2 \theta} = \frac{1 + \cot^2 \theta}{1 + \tan^2 \theta} \\
 &= \frac{\csc^2 \theta}{\sec^2 \theta} \\
 &= \frac{1}{\frac{\sin^2 \theta}{1}} \\
 &= \frac{1}{\cos^2 \theta} \\
 &= \left(\frac{1}{\sin^2 \theta} \right) \left(\frac{\cos^2 \theta}{1} \right) \\
 &= \frac{\cos^2 \theta}{\sin^2 \theta} \\
 &= \cot^2 \theta
 \end{aligned}$$