Unit 5: Trapezoidal Rule for Approximating Area under a Curve

Area of a Trapezoid = $\dfrac{(b_1 + b_2)}{2} * h$

Trapezoid Rule

where,

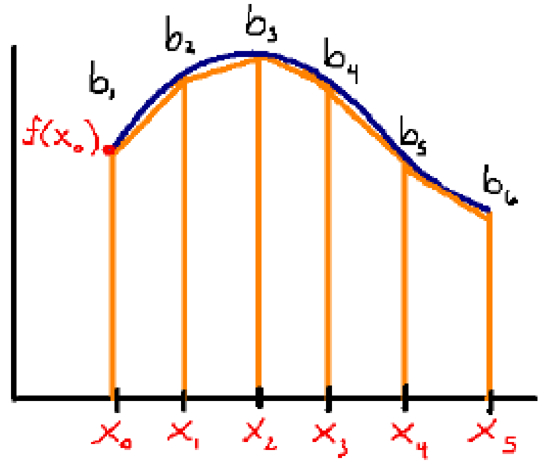
$h = \Delta x$

$b_1, b_2, \ldots = f(x_0), f(x_1)\ldots$

$b_1 = f(x_0)$

$h = \Delta x$ 

$b_2 = f(x_1)$



Approximated area using four trapezoids:

$$A = \frac{\Delta x}{2}[f(x_1) + f(x_2)] + \frac{\Delta x}{2}[f(x_2) + f(x_3)] + \frac{\Delta x}{2}[f(x_3) + f(x_4)] + \frac{\Delta x}{2}[f(x_4) + f(x_5)]$$

$$A = \frac{\Delta x}{2}[f(x_1) + f(x_2) + f(x_2) + f(x_3) + f(x_3) + f(x_4) + f(x_4) + f(x_5)]$$

$$A = \frac{\Delta x}{2}[f(x_1) + 2f(x_2) + 2f(x_3) + 2f(x_4) + f(x_5)]$$

Thus, the generalized formula for any value of n, where n is the number of trapezoids:

$$\boxed{A = \frac{\Delta x}{2}[f(x_1) + 2f(x_2) + \ldots\ldots\ldots\ldots + 2f(x_n) + f(x_{n+1})]}$$

NOTE: The trapezoidal approximation is the same as the average of the left endpoint and the right endpoint approximation using rectangles.

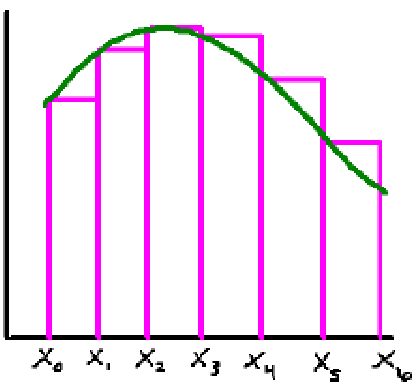
Example:

1. $f(x) = -x^2 + 4x + 1 \quad$ in $[0, 4]$ $\quad$ with $n = 4$

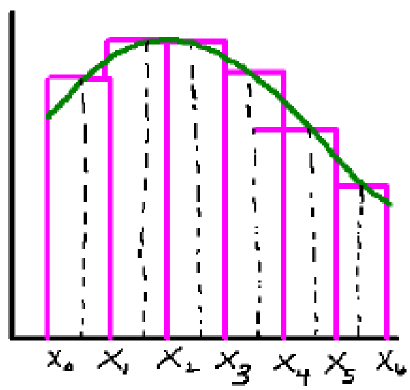
2. $f(x) = x^2 + 2$ in [-1, 2] with $n = 3$

Approximation Summary

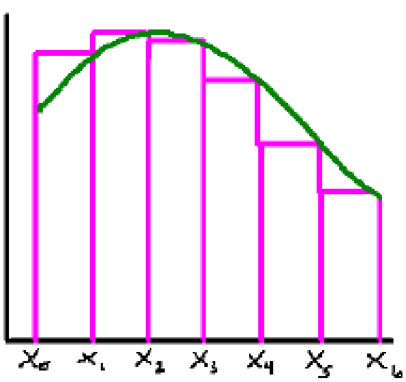
Left Endpoint Approximation



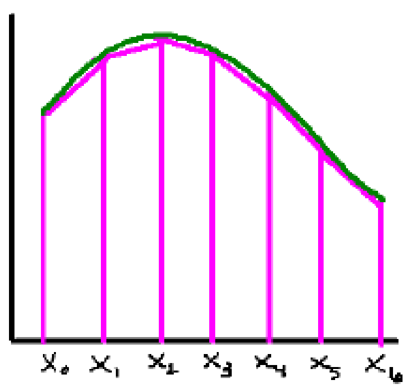
Midpoint Rule



Right Endpoint Approximation



Trapezoid Rule



Approximations with a Table of Values

The various approximation techniques, including rectangles and trapezoids, can be used to calculate the total change on a specific interval. This concept is best represented by what will be known as the:

Total Change Theorem

$$\boxed{\int_a^b f(x)dx = F(b) - F(a)}$$

, where f(x) represents a function measuring the rate of change of some quantity.

NOTE: This principle can commonly be applied to rates of change in the natural and social sciences.

Example:

The following chart indicates the speed of a sprinter during the first 10 seconds of the race.

Time (s)	0	1	3	5	6	9	10
Velocity (ft/s)	0	4.2	8.8	12.6	13.1	15.2	15.0

Approximate the value of $\int_{t=0}^{t=10} v(t)dt$ using the various approximation techniques:

a) Find the distance traveled using a left-endpoint approximation method.

b) Find the distance traveled using a right-endpoint approximation method.

c) Find the distance traveled using a trapezoidal approximation method.

<u>Applications involving the Total Change Theorem</u>

Some examples of the definite integral representing Total Change:

- If the rate at which water flows into a reservoir is by v(t), then

$\int_{t_1}^{t_2} v(t)dt = V(t_2) - V(t_1)$ **calculates to total change in Volume from t_1 to t_2.**

- If the rate of growth of a population is represented by $p(t)$, then

$\int_{t_1}^{t_2} p(t)dt = P(t_2) - P(t_1)$ **calculates the increase in population during the time period from t_1 to t_2.**

Examples: **Describe in a complete sentence each of the following scenarios presented.**

1. If *w(t)* is the rate of growth of a child's weight in pounds per year, what does $\int_{5}^{10} w(t)dt$ represent?

____

2. If oil leaks from a tank at a rate of *r(t)* gallons per minute, what does $\int_{0}^{120} r(t)dt$ represent?

____

Solve each of the following scenarios described using the total change theorem.

3. Water flows from the bottom of a storage tank at a rate of $r(t) = 200 - 4t$ liters/min, where $0 \le t \le 50$. Find the amount of water that flows from the tank during:
   a) the first 10 minutes
   b) the first half hour

4. A honeybee population starts with 100 bees and increases at a rate of $r(t) = t^2 + 2t$ bees per week. Determine the population of bees during:
   a) the end of the first month
   b) the end of the year

Using the Total Change Theorem to calculate [Displacement vs. Distance Traveled]:

NOTE: This is only relevant when examining objects that are moving in two directions.
[Up/Down] OR [Left/Right].

To find the displacement (position shift) from the velocity function, we just integrate the function. This simply calculates the change in position during the indicated time interval.

$$\text{Displacement} = \int_a^b V(t)\,dt$$

To find distance traveled from the velocity function, we have to integrate the absolute value of the function. Incorporating the absolute value will require us to know where the velocity is positive [v(t) > 0] and where the velocity is negative [v(t) < 0] on the indicated time interval. As a result, this will allow us to eliminate the direction on v(t) and integrate in pieces.

$$\text{Distance Traveled} = \int_a^b \left|V(t)\right|dt$$

Example:

5. A particle moves according to: $v(t)=49-9.8t$ during $0 \le t \le 10.$

   a. Determine when the particle is moving right, left and is stopped.

   b. Find the particle's displacement over the given time interval.

   c. Find the total distance traveled by the particle over the interval [0,10].

   d. If the initial position of the particle is at 4, find the particle's position at t=6.